The Revenue Equivalence Theorem

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Using the Envelope Theorem to Analyze Equilibrium Payoffs (1)

• The Envelope Theorem

\[ \frac{d}{dt} \max_x \pi(x, t) = \frac{d}{dt} \pi(x^*(t), t) = \frac{\partial}{\partial t} \pi(x, t) \bigg|_{x=x^*(t)} \]

• “Proof”: Recall FOC’s necessary

\[ \frac{d}{dt} \pi(x^*(t), t) = \frac{\partial}{\partial x} \pi(x, t) \bigg|_{x=x^*(t)} \frac{\partial}{\partial t} x^*(t) + \frac{\partial}{\partial t} \pi(x, t) \bigg|_{x=x^*(t)} \]
Using the Envelope Theorem to Analyze Equilibrium Payoffs (2)

• Example: How does change in MC affect monopoly profits?

\[
\frac{d}{dc} \pi(p^*(c), c) = \frac{d}{dc} \left[ (p^*(c) - c) \cdot Q^D(p^*(c)) \right]
\]

=
Using the Envelope Theorem to Analyze Equilibrium Payoffs (3)

- **Envelope Theorem in the 1st Price Auction**

\[
\pi_{FP}(v, \beta_{FP}(v)) \text{ is exp. prof. after obs. } V = v
\]

\[
\pi_{FP}(v, \beta_{FP}(v)) = (v - \beta_{FP}(v)) \Pr(\beta_{FP}(v) \text{ wins})
\]

\[
\frac{d}{dv} \pi_{FP}(v, \beta_{FP}(v)) = \frac{d}{dv} \left[ (v - \beta_{FP}(v)) \Pr(\beta_{FP}(v) \text{ wins}) \right]
\]

\[
= \Pr(\beta_{FP}(v) \text{ wins}) + \frac{d}{dv} \beta_{FP}(v) \left\{ (v - b) \frac{d}{db} \Pr(b \text{ wins}) - \Pr(b \text{ wins}) \right\}_{b=\beta_{FP}(v)}
\]

\[
= \Pr(\beta_{FP}(v) \text{ wins})
\]
Using the Envelope Theorem to Analyze Equilibrium Payoffs (3)

- Fundamental theorem of calculus

\[ g(v) - g(\underline{v}) = \int_{\underline{v}}^{v} \frac{d}{dt} g(t) dt \]

- Expected profit for bidder with value \( v \):

\[
\pi^{FP}(v, \beta^{FP}(v)) = \pi^{FP}(v, \beta^{FP}(\underline{v})) + \int_{\underline{v}}^{v} \frac{d}{dt} \pi^{FP}(t, \beta^{FP}(t)) dt \\
= 0 + \int_{\underline{v}}^{v} \text{Pr}(\beta^{FP}(t) \text{ wins}) dt
\]
Using the Envelope Theorem to Analyze Equilibrium Payoffs (3)

• Symmetric Equilibrium
  – All use same strategies
  – Probability $\beta^{FP}(v)$ wins is probability all others have value less than $v$, or $F(v)^{n-1}$

• Expected profit for bidder with value $v$:

$$\pi^{FP}(v, \beta^{FP}(v)) = \int_v^v F(t)^{n-1} dt$$
Using the Envelope Theorem to Analyze Equilibrium Payoffs (4)

• Compare to Second-Price Auction
  – Profits
  \[ \pi^{SP}(v, \beta^{SP}(v)) \]
  \[ = \Pr(\beta^{SP}(v) \text{ wins}) \cdot (v - E[\max(V_2,\ldots,V_n) | \max(V_2,\ldots,V_n) \leq \beta^{SP}(v)]) \]
  – Derivative of profits
  \[ \frac{d}{dv} \pi^{SP}(v, \beta^{SP}(v)) = \Pr(\beta^{SP}(v) \text{ wins}) = F(v)^{n-1} \]

• Expected profit for bidder with value \( v \):
  \[ \pi^{SP}(v, \beta^{SP}(v)) = \pi^{SP}(v, \beta^{SP}(v)) + \int_{v}^{\infty} F(t)^{n-1} dt = \int_{v}^{\infty} F(t)^{n-1} dt \]
Revenue Equivalence Theorem

• Theorem: In symmetric, independent private values auction model, first-price auction, second-price auction, and oral ascending auction:
  – All give the same expected profit to each bidder given $v$
  – All raise the same expected revenue for the seller
Interpretations

- Seems like first-price auction should raise more revenue
  - Pay your bid, not 2\textsuperscript{nd}-highest bid
  - But, in equilibrium, shade your bid
- Why does bid shading just equal out with FPA v. SPA?
  - Only way different types differ is that higher types value winning more; both value money the same
  - Only way equilibrium utility can change with value is that different types have different probabilities of winning
  - In symmetric eqm, highest value wins in all formats
Interpretations

• Critical factors for applying theorem
  – Symmetry: all auction formats have same probability of winning
  – Breaks down with asymmetric distributions since in FPA optimal to bid less aggressively against weak bidders

• Misleading: SPA is “better” because bidders tell truth
  – Easier to analyze
  – More robust: don’t have to calculate probabilities
  – But, cannot get bidders to pay full value
Application: Using the RET to Solve for Bidding Function

• RET:
  – Profit to a bidder with value $v$ is the same in FPA and SPA

$$\pi_{FP}^{FP}(v, \beta_{FP}^{FP}(v)) = \pi_{SP}^{SP}(v, \beta_{SP}^{SP}(v))$$

$$F(v)^{n-1} \cdot (v - \beta_{FP}^{FP}(v)) = F(v)^{n-1} \cdot (v - E[V^{n-1:n} | V^{n:n} = v])$$

• Back out bid function

$$\beta_{FP}^{FP}(v) = E[V^{n-1:n} | V^{n:n} = v]$$
Using the RET to Solve for Bidding Function: Uniform Example

• Example
  – n bidders
  – Each draws value from U[0,1]
  – Properties of Uniform:
    \[ F(v)^{n-1} = v^{n-1} \]
    \[ E[V^{n-1:n}] = \frac{n-1}{n+1} \]
    \[ E[V^{n-1:n} | V^{n:n} = v] = \frac{n-1}{n} v \]

• Questions
  – What is expected payment, conditional on winning, for a bidder with value \( v \) in a second-price auction?
  – What is equilibrium bid in a first-price auction?
  – What is expected revenue for the seller in either auction?
Applications

• eBay bidding
  – Is it optimal to just use proxy bidding as eBay suggests?
  – Why bother with an ascending auction?
    • Alternative: second-price, sealed bid, advertised for fixed length of time