Math212a Lecture 5
Applications of the spectral theorem for compact self-adjoint operators, 2.
Gårdings inequality and its consequences.

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1. **Review of Sobolev spaces.**
   - Distributions and Schwartz's theorem.

2. **Gårding's inequality.**
   - Differential operators.
   - Rellich's lemma
   - Some numerical inequalities.
   - Elliptic operators.
   - Statement and proof of Gårding's inequality.

3. **Consequences of Gårding's inequality.**

4. **Extension of the basic lemmas to manifolds.**
   - Example: Hodge theory.

5. **The resolvent.**
The space $P(T)$ and its scalar products.

Recall that $T$ now stands for the $n$-dimensional torus. Let $P = P(T)$ denote the space of trigonometric polynomials. These are functions on the torus of the form

$$u(x) = \sum a_\ell e^{i\ell \cdot x}$$

where $\ell = (\ell_1, \ldots, \ell_n)$ is an $n$-tuplet of integers and the sum is finite. For each integer $t$ (positive, zero or negative) we introduced the scalar product

$$(u, v)_t := \sum_\ell (1 + \ell \cdot \ell)^t a_\ell \overline{b_\ell}. \quad (1)$$
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For \(t = 0\) this is the scalar product

\[(u, v)_0 = \frac{1}{(2\pi)^n} \int_{\mathbb{T}} u(x)\overline{v(x)}dx.\]
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$$ (u, v)_0 = \frac{1}{(2\pi)^n} \int_{\mathbb{T}} u(x)\overline{v(x)} \, dx. $$

We denote the norm corresponding to the scalar product $(\cdot, \cdot)_s$ by $\| \cdot \|_s$. 
Relations between the norms.

If

\[ \Delta := - \left( \frac{\partial^2}{\partial (x^1)^2} + \cdots + \frac{\partial^2}{\partial (x^n)^2} \right) \]

the operator \((1 + \Delta)\) satisfies

\[ (1 + \Delta)u = \sum (1 + \ell \cdot \ell) a_\ell e^{i\ell \cdot x} \]

and so

\[ ((1 + \Delta)^t u, \nu)_s = (u, (1 + \Delta)^t \nu)_s = (u, \nu)_{s+t} \]

and

\[ \|(1 + \Delta)^t u\|_s = \|u\|_{s+2t}. \quad (2) \]
The generalized Cauchy Schwarz inequality.

We then get the \textbf{generalized Cauchy-Schwarz inequality}

\[ |(u, v)_s| \leq \|u\|_{s+t} \|v\|_{s-t} \quad (3) \]

for any $t$, as a consequence of the usual Cauchy-Schwarz inequality.
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\[ |(u, v)_s| \leq \|u\|_{s+t}\|v\|_{s-t} \]  \quad (3)

for any \( t \), as a consequence of the usual Cauchy-Schwarz inequality. Indeed,

\[
\sum_{\ell} (1 + \ell \cdot \ell)^s a_\ell \overline{b}_\ell = \sum_{\ell} (1 + \ell \cdot \ell)^{\frac{s+t}{2}} a_\ell (1 + \ell \cdot \ell)^{\frac{s-t}{2}} \overline{b}_\ell \\
= \langle (1 + \Delta)^{\frac{s+t}{2}} u, (1 + \Delta)^{\frac{s-t}{2}} v \rangle_0 \\
\leq \|(1 + \Delta)^{\frac{s+t}{2}} u\|_0 \|(1 + \Delta)^{\frac{s-t}{2}} v\|_0 \\
= \|u\|_{s+t}\|v\|_{s-t}.
\]
The generalized Cauchy-Schwarz inequality reduces to the usual Cauchy-Schwarz inequality when $t = 0$. 
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Clearly we have

$$\|u\|_s \leq \|u\|_t \quad \text{if } s \leq t.$$ 

If $D^p$ denotes a partial derivative,

$$D^p = \frac{\partial |p|}{\partial (x^1)^{p_1} \cdots \partial (x^n)^{p_m}}$$

then

$$D^p u = \sum (i\ell)^p a_\ell e^{i\ell \cdot x}.$$
In these equations we are using the following notations:

- If $p = (p_1, \ldots, p_n)$ is a vector with non-negative integer entries we set

$$|p| := p_1 + \cdots + p_n.$$
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- If $\xi = (\xi_1, \ldots, \xi_n)$ is a (row) vector we set
  $$\xi^p := \xi_1^{p_1} \cdot \xi_2^{p_2} \cdots \xi_n^{p_n}.$$
It is then clear that

\[ \| D^p u \|_t \leq \| u \|_{t+|p|} \] (4)

and similarly

\[ \| u \|_t \leq (\text{constant depending on } t) \sum_{|p| \leq t} \| D^p u \|_0 \quad \text{if } t \geq 0. \] (5)

In particular,
Theorem

The norms

\[ u \mapsto \| u \|_t \]

\[ t \geq 0 \text{ and } \]

\[ u \mapsto \sum_{|p| \leq t} \| D^p u \|_0 \]

are equivalent.
The Sobolev spaces $H_t$.

We let $H_t$ denote the completion of the space $P$ with respect to the norm $\| \cdot \|_t$. Each $H_t$ is a Hilbert space, and we have natural embeddings

$$H_t \hookrightarrow H_s \quad \text{if} \quad s < t.$$ 

The equation

$$\|(1 + \Delta)^t u\|_s = \|u\|_{s+2t}$$

says that $(1 + \Delta)^t$, initially defined on $P$, extends to a map

$$(1 + \Delta)^t : H_{s+2t} \to H_s$$

and is an isometry.
The duality between $H_t$ and $H_{-t}$.

From the generalized Cauchy-Schwartz inequality we also have a natural pairing of $H_t$ with $H_{-t}$ given by the extension of $(\ , \ )_0$, so

$$|(u, v)_0| \leq \|u\|_t \|v\|_{-t}. \quad (6)$$

In fact, this pairing allows us to identify $H_{-t}$ with the space of continuous linear functions on $H_t$. Let us state this in more detail:
Theorem

Let \( v \in H_{-t} \). Then \( v \) defines a continuous linear function \( \phi_v \) on \( H_t \) by

\[
\phi_v(u) = (u, v)_0
\]

and \( \| \phi_v \| = \| v \|_{-t} \), i.e.

\[
\| v \|_{-t} = \sup \| (u, v)_0 \|, \quad \| u \|_t = 1.
\]

Conversely, every continuous linear function \( \phi \) on \( H_t \) is of the form \( \phi_v \) for a unique \( v \in H_{-t} \).
Proof.

If $\phi$ is a continuous linear function on $H_t$ the Riesz representation thm says that there is a unique $w \in H_t$ with $\phi(u) = (u, w)_t$ and

$$\|\phi\| = \sup_{\|u\|_t = 1} |(u, w)_t| = \|w\|_t.$$

Set

$$v := (1 + \Delta)^t w.$$

Then $v \in H_{-t}$ and

$$(u, v)_0 = (u, (1 + \Delta)^t w)_0 = (u, w)_t = \phi(u).$$

Starting with $v \in H_{-t}$ we get the continuous linear function $\phi : u \mapsto (u, v)_0$ with $\phi = \phi_v$ (and $w = (1 + \Delta)^{-t} v$).
We record the theorem as
\[ H_{-t} = (H_t)^* . \tag{7} \]

As an illustration of (7), observe that the series
\[ \sum_{\ell} (1 + \ell \cdot \ell)^s \]
converges for \( s < -\frac{n}{2} \). This means that if define \( v \) by taking
\[ b_\ell \equiv 1 \]
then \( v \in H_s \) for \( s < -\frac{n}{2} \).
The Dirac delta function.

This means that if define \( v \) by taking

\[
b_\ell \equiv 1
\]

then \( v \in H_s \) for \( s < -\frac{n}{2} \).

If \( u \) is given by \( u(x) = \sum_\ell a_\ell e^{i\ell \cdot x} \) is any trigonometric polynomial, then

\[
(u, v)_0 = \sum a_\ell = u(0).
\]

So the natural pairing (6) allows us to extend the linear function sending \( u \mapsto u(0) \), initially defined only on \( P(T) \), to all of \( H_t \) if \( t > \frac{n}{2} \). We can now give \( v \) its “true name”: it is the Dirac “delta function” \( \delta \) (on the torus) where

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(u, \delta)_0 = u(0).
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So \( \delta \in H_{-t} \) for \( t > \frac{n}{2} \), and the preceding equation is usually written symbolically as
\[
\frac{1}{(2\pi)^n} \int_{\mathbb{T}} u(x)\delta(x)dx = u(0);
\]

but the true mathematical interpretation is as given above.
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We set

\[
H_{\infty} := \bigcap H_t, \quad H_{-\infty} := \bigcup H_t.
\]
Sobolev’s Lemma.

The space $H_0$ is just $L_2(\mathbb{T})$, and we can think of the space $H_t$, $t > 0$ as consisting of those functions having “generalized $L_2$ derivatives up to order $t$”. Certainly a function of class $C^t$ belongs to $H_t$. With a loss of degree of differentiability the converse is true:

**Lemma**

[Sobolev.] If $u \in H_t$ and

$$t \geq \left[\frac{n}{2}\right] + k + 1$$

then $u \in C^k(\mathbb{T})$ and

$$\sup_{x \in \mathbb{T}} |D^p u(x)| \leq \text{const.} \|u\|_t \quad \text{for } |p| \leq k.$$  (8)
Distributions aka generalized functions.

A **distribution** on $\mathbb{T}^n$ is a linear function $T$ on $C^\infty(\mathbb{T}^n)$ with the continuity condition that

$$\langle T, \phi_k \rangle \to 0$$

whenever

$$D^p \phi_k \to 0$$

uniformly for each fixed $p$. If $u \in H_{-t}$ we may define

$$\langle u, \phi \rangle := (\phi, \overline{u})_0$$

and since $C^\infty(\mathbb{T})$ is dense in $H_t$ we may conclude
Schwartz’s theorem.

Lemma

$H_{-t}$ is the space of those distributions $T$ which are continuous in the $\| \|_t$ norm, i.e. which satisfy

$$\| \phi_k \|_t \to 0 \quad \Rightarrow \quad \langle T, \phi_k \rangle \to 0.$$  

We then obtain

Theorem

[Laurent Schwartz.] $H_{-\infty}$ is the space of all distributions. In other words, any distribution belongs to $H_{-t}$ for some $t$. 
Multiplication by a smooth function.

Suppose that $f$ is a $C^\infty$ function on $\mathbb{T}$. Multiplication by $f$ is clearly a bounded operator on $C^\infty(\mathbb{T})$ in the $L_2$ norm and so extends to a bounded operator on $H_0 = L_2(\mathbb{T})$. Similarly, it extends to a bounded operator on $H_t$, $t > 0$ since we can expand $D^p(fu)$ by applications of Leibnitz’s rule for $u \in C^\infty(\mathbb{T})$.

For $t = -s < 0$ we know by our theorem that

$$
\|fu\|_t = \sup |(v, fu)_0|/\|v\|_s = \sup |(u, \bar{f}v)_0|/\|v\|_s \leq \|u\|_t \sup \|\bar{f}v\|_s/\|v\|_s.
$$

So in all cases we have

$$
\|fu\|_t \leq (\text{const. depending on } f \text{ and } t) \|u\|_t. \tag{9}
$$
Let

$$L = \sum_{|p| \leq m} \alpha_p(x) D^p$$

be a differential operator of degree $m$ with $C^\infty$ coefficients. Then it follows from the above that

$$\|Lu\|_{t-m} \leq \text{constant}\|u\|_t \quad (10)$$

where the constant depends on $L$ and $t$. 
Rellich’s lemma

**Lemma**

*Rellich’s lemma.* If $s < t$ the embedding $H_t \hookrightarrow H_s$ is compact.

We must show that the image of the unit ball $B$ of $H_t$ in $H_s$ can be covered by finitely many balls of radius $\varepsilon$.

To start the proof, Choose $N$ so large that
\[(1 + \ell \cdot \ell)^{(s-t)/2} < \frac{\varepsilon}{2}\]
when $\ell \cdot \ell > N$. 

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Proof.

Let $Z_t$ be the subspace of $H_t$ consisting of all $u$ such that $a_\ell = 0$ when $\ell \cdot \ell \leq N$. 

This is a space of finite codimension, and hence the unit ball of $Z_t^\perp \subset H_t$ can be covered by finitely many balls of radius $\epsilon^2$. The space $Z_t^\perp$ consists of all $u$ such that $a_\ell = 0$ when $\ell \cdot \ell > N$. The image of $Z_t^\perp$ in $H_s$ is the orthogonal complement of the image of $Z_t$.

On the other hand, for $u \in B \cap Z_t$ we have $\|u\|_2^2 \leq (1 + N)^{s-t} \|u\|_t^2 \leq (\epsilon^2)^2$. So the image of $B \cap Z_t$ is contained in a ball of radius $\epsilon^2$. Every element of the image of $B$ can be written as a (orthogonal) sum of an element in the image of $B \cap Z_t^\perp$ and an element of $B \cap Z_t$ and so the image of $B$ is covered by finitely many balls of radius $\epsilon^2$. 

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**Proof.**

Let $Z_t$ be the subspace of $H_t$ consisting of all $u$ such that $a_\ell = 0$ when $\ell \cdot \ell \leq N$. This is a space of finite codimension, and hence the unit ball of $Z_t^\perp \subset H_t$ can be covered by finitely many balls of radius $\frac{\epsilon}{2}$. The space $Z_t^\perp$ consists of all $u$ such that $a_\ell = 0$ when $\ell \cdot \ell > N$. The image of $Z_t^\perp$ in $H_s$ is the orthogonal complement of the image of $Z_t$. 
Proof.

Let $Z_t$ be the subspace of $H_t$ consisting of all $u$ such that $a_\ell = 0$ when $\ell \cdot \ell \leq N$. This is a space of finite codimension, and hence the unit ball of $Z_t^\perp \subset H_t$ can be covered by finitely many balls of radius $\frac{\epsilon}{2}$. The space $Z_t^\perp$ consists of all $u$ such that $a_\ell = 0$ when $\ell \cdot \ell > N$. The image of $Z_t^\perp$ in $H_s$ is the orthogonal complement of the image of $Z_t$. On the other hand, for $u \in B \cap Z_t$ we have

$$\|u\|_s^2 \leq (1 + N)^{s-t}\|u\|_t^2 \leq \left(\frac{\epsilon}{2}\right)^2.$$ 

So the image of $B \cap Z_t$ is contained in a ball of radius $\frac{\epsilon}{2}$. Every element of the image of $B$ can be written as a (n orthogonal) sum of an element in the image of $B \cap Z_t^\perp$ and an element of $B \cap Z_t$ and so the image of $B$ is covered by finitely many balls of radius $\epsilon$. \qed
Some numerical inequalities.

A useful numerical inequality.

Let $x > 0$ be a positive number, and $a$ and $b$ be non-negative numbers. Then

$$x^a + x^{-b} \geq 1$$

because if $x \geq 1$ the first summand is $\geq 1$ and if $x \leq 1$ the second summand is $\geq 1$. Setting $x = \epsilon^{1/a} A$ gives

$$1 \leq \epsilon A^a + \epsilon^{-b/a} A^{-b}$$

if $\epsilon$ and $A$ are positive.
A consequence.

\[ 1 \leq \epsilon A^a + \epsilon^{-b/a} A^{-b} \]

if \(\epsilon\) and \(A\) are positive. Suppose that \(t_1 > s > t_2\) and we set \(a = t_1 - s, \ b = s - t_2\) and \(A = 1 + \ell \cdot \ell\). Then we get

\[ (1 + \ell \cdot \ell)^s \leq \epsilon (1 + \ell \cdot \ell)^{t_1} + \epsilon^{-(s-t_2)/(t_1-s)}(1 + \ell \cdot \ell)^{t_2} \]

and therefore

\[ \|u\|_s \leq \epsilon \|u\|_{t_1} + \epsilon^{-(s-t_2)/(t_1-s)} \|u\|_{t_2} \quad \text{if} \quad t_1 > s > t_2, \ \epsilon > 0 \quad (11) \]

for all \(u \in H_{t_1}\). This elementary inequality will be the key to several arguments in today’s lecture where we will combine (11) with integration by parts.
Some numerical inequalities.

$$\| u \|_s \leq \epsilon \| u \|_{t_1} + \epsilon^{-(s-t_2)/(t_1-s)} \| u \|_{t_2} \text{ if } t_1 > s > t_2, \ \epsilon > 0.$$  \hspace{1cm} (11)

We may sometimes refer to (11) as the “little constant - big constant” inequality. It says that we can estimate $\| u \|_s$ in terms of a small constant times $\| u \|_{t_1}$ for $t_1 > s$ provided we add a large constant times $\| u \|_{t_2}$ for $t_2 < s.$
Elliptic differential operators.

A differential operator \( L = \sum_{|p| \leq m} a_p(x) D^p \) with real coefficients and \( m \) even is called **elliptic** if there is a constant \( c > 0 \) such that

\[
(-1)^{m/2} \sum_{|p|=m} a_p(x) \xi^p \geq c (\xi \cdot \xi)^{m/2} \quad \forall \, x, \, \xi.
\]

For example, the operator \( \Delta := - \left( \frac{\partial^2}{\partial (x_1)^2} + \cdots + \frac{\partial^2}{\partial (x_n)^2} \right) \) is elliptic. The vector \( \xi \) in (12) is a “dummy variable”.
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\]

For example, the operator \( \Delta := -\left(\frac{\partial^2}{\partial(x^1)^2} + \cdots + \frac{\partial^2}{\partial(x^n)^2}\right) \) is elliptic. The vector \( \xi \) in (12) is a “dummy variable”. (Its true significance is that it is a covector, i.e. an element of the cotangent space at \( x \).)
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A differential operator $L = \sum_{|p| \leq m} a_p(x)D^p$ with real coefficients and $m$ even is called **elliptic** if there is a constant $c > 0$ such that

$$(-1)^{m/2} \sum_{|p|=m} a_p(x)\xi^p \geq c(\xi \cdot \xi)^{m/2} \quad \forall \ x, \xi. \quad (12)$$

For example, the operator $\Delta := -\left(\frac{\partial^2}{\partial(x^1)^2} + \cdots + \frac{\partial^2}{\partial(x^n)^2}\right)$ is elliptic. The vector $\xi$ in (12) is a “dummy variable”. (Its true significance is that it is a covector, i.e. an element of the cotangent space at $x$.) The expression on the left of (12) is called the **symbol** of the operator $L$. It is a homogeneous polynomial of degree $m$ in the variable $\xi$ whose coefficients are functions of $x$. 
The symbol of $L$ is sometimes written as $\sigma(L)$ or $\sigma(L)(x, \xi)$. Another way of expressing the ellipticity condition (12) is:

There is a positive constant $c$ such that

$$\sigma(L)(x, \xi) \geq c \text{ for all } x \text{ and } \xi \text{ such that } \xi \cdot \xi = 1.$$
We will assume until further notice that the operator $L$ is elliptic and that $m$ is a positive even integer.

**Theorem**

[Gårding’s inequality.] For every $u \in C^\infty(T)$ we have

$$(u, Lu)_0 \geq c_1 \|u\|_{m/2}^2 - c_2 \|u\|_0^2$$

(13)

where $c_1 > 0$ and $c_2$ are constants depending on $L$.

**Remark.** If $u \in H_{m/2}$, both sides of (13) make sense, and we can approximate $u$ in the $\| \|_{m/2}$ norm by $C^\infty$ functions. So once we prove the theorem, we conclude that it is true for all of $H_{m/2}$.
Outline of steps in the proof of Gårding’s inequality.

We will prove the theorem in stages:

1. When \( L \) is constant coefficient and homogeneous.
2. When \( L \) is homogeneous and approximately constant.
3. When the \( L \) can have lower order terms but the (top order) homogeneous part of \( L \) is approximately constant.
4. The general case.
Statement and proof of Gårding's inequality.

**Stage 1: \( L \) is constant coefficient and homogeneous.**

\[ L = \sum_{|p|=m} \alpha_p D^p \] where the \( \alpha_p \) are constants. Then

\[
(u, Lu)_0 = \left( \sum \alpha_p (i \ell)^p \right) a_\ell e^{i \ell \cdot x}
\]

\[ \geq c \sum (\ell \cdot \ell)^{m/2} |a_\ell|^2 \text{ by (12)} \]

\[ = c \sum [1 + (\ell \cdot \ell)^{m/2} |a_\ell|^2 - c \|u\|^2_0 \geq cC \|u\|^{2} - c \|u\|_0 \]

where

\[ C = \inf_{r \geq 0} \frac{1 + r^{m/2}}{(1 + r)^{m/2}} > 0. \]

This takes care of stage 1.
Stage 2.

Here \(L = L_0 + L_1\) where \(L_0\) is as in stage 1 and 
\[L_1 = \sum_{|p|=m} \beta_p(x)D^p\] and

\[
\max_{p,x} |\beta_p(x)| < \eta,
\]

where \(\eta\) sufficiently small. (How small will be determined very soon in the course of the discussion.) We have

\[
(u, L_0u)_0 \geq c'\|u\|_{m/2}^2 - c\|u\|_0^2
\]

from stage 1.
We integrate \((u, L_1 u)_0\) by parts \(m/2\) times. There are no boundary terms since we are on the torus. In integrating by parts some of the derivatives will hit the coefficients. Let us collect all the these terms as \(I_2\). The other terms we collect as \(I_1\), so

\[
I_1 = \sum \int b_{p'} + p'' D^{p'} u \overline{D^{p''} u} \, dx
\]

where \(|p'| = |p''| = m/2\) and \(b_r = \pm \beta_r\). We can estimate this sum by

\[
|I_1| \leq \eta \cdot \text{const.} \|u\|^2_{m/2}
\]

and so will require that \(\eta \cdot (\text{const.}) < c'\).
The remaining terms give a sum of the form

\[ l_2 = \sum \int b_{p'q} D^{p'} u \overline{D^q u} \, dx \]

where \( p' \leq m/2, \, q' < m/2 \) so we have

\[ |l_2| \leq \text{const.} \|u\|_{m/2} \|u\|_{m/2-1}. \]

Recall the “little constant big constant inequality”:

\[ \|u\|_s \leq \epsilon \|u\|_{t_1} + \epsilon^{-(s-t_2)/(t_1-s)} \|u\|_{t_2} \quad \text{if } t_1 > s > t_2, \quad \epsilon > 0 \] (11).

Take \( s = \frac{m}{2} - 1, \, t_1 = \frac{m}{2}, \, t_2 = 0 \) yielding, for any \( \epsilon > 0 \),

\[ \|u\|_{m/2-1} \leq \epsilon \|u\|_{m/2} + \epsilon^{1-\frac{m}{2}} \|u\|_0. \]
We have

\[ |l_2| \leq \text{const.} \|u\|^{m/2} \|u\|^{m/2-1} \]

and

\[ \|u\|^{m-1/2} \leq \epsilon \|u\|^{m/2} + \epsilon^{1-\frac{m}{2}} \|u\|_0. \]

Substituting this inequality into the above estimate for \( l_2 \) gives

\[ |l_2| \leq \epsilon \cdot \text{const.} \|u\|^{2} \|u\|_{m/2} + \epsilon^{1-\frac{m}{2}} \text{const.} \|u\|_{m/2} \|u\|_0. \]
Statement and proof of Gårding’s inequality.

\[ |l_2| \leq \epsilon \cdot \text{const.} \|u\|_{m/2}^2 + \epsilon^{1-m/2} \text{const.} \|u\|_{m/2} \|u\|_0. \]

For any positive numbers \( a, b \) and \( \zeta \) the inequality \( (\zeta a - \zeta^{-1} b)^2 \geq 0 \) implies that \( 2ab \leq \zeta^2 a^2 + \zeta^{-2} b^2 \). Taking \( \zeta^2 = \epsilon^{m+1} \) we can replace the second term on the right in the preceding estimate for \( |l_2| \) by

\[ \frac{1}{2} \left[ \epsilon^2 \|u\|_{m/2}^2 + \epsilon^{-m} \|u\|_0^2 \right]. \]

We can absorb the \( \frac{1}{2} \epsilon \) into the constant appearing in the first term in the estimate at the top of the screen.
We have thus established that

\[ |l_1| \leq \eta \cdot (\text{const.})_1 \|u\|_{m/2}^2 \]

where the constant depends only on \( m \), and

\[ |l_2| \leq \epsilon (\text{const.})_2 \|u\|_{m/2}^2 + \epsilon^{-m} \text{const.} \|u\|_0^2 \]

where the constants depend on \( L_1 \) but \( \epsilon \) is at our disposal.
We have thus established that

\[ |l_1| \leq \eta \cdot (\text{const.})_1 \|u\|^{2}_{m/2} \]

where the constant depends only on \( m \), and

\[ |l_2| \leq \epsilon (\text{const.})_2 \|u\|^{2}_{m/2} + \epsilon^{-m} \text{const.} \|u\|^{2}_0 \]

where the constants depend on \( L_1 \) but \( \epsilon \) is at our disposal. So if \( \eta (\text{const.})_1 < c' \) and we then choose \( \epsilon \) so that \( \epsilon (\text{const.})_2 < c' - \eta \cdot (\text{const.})_1 \) we obtain Gårding’s inequality for this case.
Stage 3.

Here $L = L_0 + L_1 + L_2$ where $L_0$ and $L_1$ are as in stage 2, and $L_2$ is a lower order operator. Here we integrate by parts and argue as in stage 2.
Choose an open covering of $\mathbb{T}$ such that the variation of each of the highest order coefficients in each open set is less than the $\eta$ of stage 2. (Recall that this choice of $\eta$ depended only on $m$ and the $c$ that entered into the definition of ellipticity.) Thus, if $v$ is a smooth function supported in one of the sets of our cover, the action of $L$ on $v$ is the same as the action of an operator as in case 3) on $v$, and so we may apply Gårding’s inequality. Choose a finite subcover and a partition of unity $\{\phi_i\}$ subordinate to this cover. Write $\phi_i = \psi_i^2$ (where we choose the $\phi$ so that the $\psi$ are smooth). So $\sum \psi_i^2 \equiv 1$. 

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Statement and proof of Gårding’s inequality.

So $\sum \psi_i^2 \equiv 1$. Now $\langle \psi_i u, L(\psi_i u) \rangle_0 \geq c'' \|\psi_i u\|_{m/2}^2 - \text{const.} \|\psi_i u\|_0^2$ where $c''$ is a positive constant depending only on $c, \eta$, and on the lower order terms in $L$. We have

$$(u, Lu)_0 = \int (\sum \psi_i^2 u \overline{Lu} \, dx) = \sum \langle \psi_i u, L\psi_i u \rangle_0 + R$$

where $R$ involves derivatives of the $\psi_i$ and hence lower order derivatives of $u$. These can be estimated as in case 2) above, and so we get

$$(u, Lu)_0 \geq c''\| \psi_i u \|_{m/2}^2 - \text{const.} \| u \|_0^2$$

(14)

since $\| \psi_i u \|_0 \leq \| u \|_0$. 
Statement and proof of Gårding’s inequality.

\[(u, Lu)_0 \geq c''' \sum \| \psi_i u \|_{m/2}^2 - \text{const.} \| u \|_0^2 \quad (14)\]

since \( \| \psi_i u \|_0 \leq \| u \|_0 \). Now \( \| u \|_{m/2} \) is equivalent, as a norm, to \( \sum_{p \leq m/2} \| D^p u \|_0 \) as we verified in the last lecture. Also

\[
\sum \| D^p (\psi_i u) \|_0^2 = \sum \| \psi_i D^p u \|_0^2 + R'
\]

where \( R' \) involves terms differentiating the \( \psi \) and so lower order derivatives of \( u \). Hence

\[
\sum \| \psi_i u \|_{m/2}^2 \geq \text{pos. const.} \| u \|_{m/2}^2 - \text{const.} \| u \|_0^2
\]

by the integration by parts argument again.
Statement and proof of Gårding’s inequality.

Hence by (14)

\[(u, Lu)_0 \geq c''' \sum \| \psi_i u \|^2_{m/2} - \text{const.} \| u \|^2_0 \]

\[\geq \text{pos. const.} \| u \|^2_{m/2} - \text{const.} \| u \|^2_0\]

which is Gårding’s inequality. \(\square\)
For the time being we will continue to study the case of the torus. But a look ahead is in order. In this last step of the argument, where we applied the partition of unity argument, we have really freed ourselves of the restriction of being on the torus. Once we make the appropriate definitions, we will then get Gårding’s inequality for elliptic operators on manifolds. Furthermore, the consequences we are about to draw from Gårding’s inequality will be equally valid in the more general setting.
Outline

- Review of Sobolev spaces
- Gårding’s inequality
- Consequences of Gårding’s inequality
- Extension of the basic lemmas to manifolds
- The resolvent

Statement and proof of Gårding’s inequality.

Lars Gårding

Born in 1919
Lemma

For every integer \( t \) there is a constant \( c(t) = c(t, L) \) and a positive number \( \Lambda = \Lambda(t, L) \) such that

\[
\|u\|_t \leq c(t) \|Lu + \lambda u\|_{t-m}
\]

when

\[ \lambda > \Lambda \]

for all smooth \( u \), and hence for all \( u \in \mathbf{H}_t \).
**Lemma**

*For every integer \( t \) there is a constant \( c(t) = c(t, L) \) and a positive number \( \Lambda = \Lambda(t, L) \) such that*

\[
\|u\|_t \leq c(t)\|Lu + \lambda u\|_{t-m} \quad (15)
\]

*when*

\[\lambda > \Lambda\]

*for all smooth \( u \), and hence for all \( u \in H_t \).*

**Proof.** Let \( s \) be some non-negative integer. We will first prove (15) for \( t = s + \frac{m}{2} \).
We have

\[ \|u\|_t \|Lu + \lambda u\|_{t-m} = \|u\|_t \|Lu + \lambda u\|_{s-m/2} \]

\[ = \|u\|_t \|(1 + \Delta)^s Lu + \lambda (1 + \Delta)^s u\|_{s-m/2} \]

\[ \geq |(u, (1 + \Delta)^s Lu + \lambda (1 + \Delta)^s u)_0| \]

by the generalized Cauchy - Schwarz inequality.
The operator \((1 + \Delta)^s L\) is elliptic of order \(m + 2s\) so

\[
\|(1 + \Delta)^t u\|_s = \|u\|_{s+2t}
\]

and Gårding’s inequality gives

\[
(u, (1 + \Delta)^s Lu + \lambda(1 + \Delta)^s u)_0 \geq c_1 \|u\|^2_{s+\frac{m}{2}} - c_2 \|u\|^2_0 + \lambda \|u\|^2_s.
\]

Since \(\|u\|_s \geq \|u\|_0\) we can combine the two previous inequalities to get

\[
\|u\|_t \|Lu + \lambda u\|_{t-m} \geq c_1 \|u\|^2_t + (\lambda - c_2) \|u\|^2_0.
\]

If \(\lambda > c_2\) we can drop the second term and divide by \(\|u\|_t\) to obtain (15).
We now prove the lemma for the case \( t = \frac{m}{2} - s \) by the same sort of argument: We have

\[
\| u \|_t \| Lu + \lambda u \|_{-s-m} = \| (1 + \Delta)^{-s} u \|_{s+m} \| Lu + \lambda u \|_{-s-m} 
\]

\[
\geq ((1 + \Delta)^{-s} u, L(1 + \Delta)^s (1 + \Delta)^{-s} u + \lambda u)_0.
\]

Now use the fact that \( L(1 + \Delta)^s \) is elliptic of order \( m + 2s \) and \( \text{Gårding’s inequality} \) to continue the above inequalities as

\[
\geq c_1 \| (1 + \Delta)^{-s} u \|_{s+m}^2 - c_2 \| (1 + \Delta)^{-s} u \|_0^2 + \lambda \| u \|_{-s}^2
\]

\[
= c_1 \| u \|_t^2 - c_2 \| u \|_{-2s}^2 + \lambda \| u \|_{-s}^2 \geq c_1 \| u \|_t^2
\]

if \( \lambda > c_2 \). Again we may then divide by \( \| u \|_t \) to get the result. \( \Box \)
\[ \|u\|_t \leq c(t)\|Lu + \lambda u\|_{t-m}. \]  \hspace{1cm} (15)

The operator \( L + \lambda I \) is a bounded operator from \( H_t \) to \( H_{t-m} \) (for any \( t \)). Suppose we fix \( t \) and choose \( \lambda \) so large that (15) holds. Then (15) says that \( L + \lambda I \) is invertible on its image, and this inverse is bounded there with a bound independent of \( \lambda > \Lambda \).
\[ \|u\|_t \leq c(t)\|Lu + \lambda u\|_{t-m}. \quad (15) \]

The operator \( L + \lambda I \) is a bounded operator from \( H_t \) to \( H_{t-m} \) (for any \( t \)). Suppose we fix \( t \) and choose \( \lambda \) so large that (15) holds. Then (15) says that \( (L + \lambda I) \) is invertible on its image, and this inverse is bounded there with a bound independent of \( \lambda > \Lambda \).

The boundedness of \( (L + \lambda I)^{-1} \) on its image, implies that this image is a closed subspace of \( H_{t-m} \).
\[ \|u\|_t \leq c(t)\|Lu + \lambda u\|_{t-m}. \quad (15) \]

The operator \( L + \lambda I \) is a bounded operator from \( H_t \) to \( H_{t-m} \) (for any \( t \)). Suppose we fix \( t \) and choose \( \lambda \) so large that (15) holds. Then (15) says that \( (L + \lambda I) \) is invertible on its image, and this inverse is bounded there with a bound independent of \( \lambda > \Lambda \).

The boundedness of \( (L + \lambda I)^{-1} \) on its image, implies that this image is a closed subspace of \( H_{t-m} \). Indeed, if \( v_n = (L + \lambda I)u_n \) and \( v_n \to v \) then the \( v_n \) form a Cauchy sequence and hence so do the \( u_n \). So \( u_n \to u \) and we conclude that \( v = (L + \lambda I)u \).
Let us show that this image is all of $H_{t-m}$ for $\lambda$ large enough. Suppose not, which means that there is some $w \in H_{t-m}$ with

$$(w, Lu + \lambda u)_{t-m} = 0$$

for all $u \in H_t$. We can write this last equation as

$$((1 + \Delta)^{t-m} w, Lu + \lambda u)_0 = 0.$$
\[(1 + \Delta)^{t-m}w, Lu + \lambda u)_0 = 0.\]

Integration by parts gives the adjoint differential operator \(L^*\) characterized by

\[(\phi, L\psi)_0 = (L^*\phi, \psi)_0\]

for all smooth functions \(\phi\) and \(\psi\), and by passing to the limit this holds for all \(\phi \in H_s, \psi \in H_t\) with \(s + t \geq m\). The operator \(L^*\) has the same leading term as \(L\) and hence is elliptic.
\[ \|u\|_t \leq c(t) \|Lu + \lambda u\|_{t-m}. \quad (15) \]

\[ ((1 + \Delta)^{t-m}w, Lu + \lambda u)_0 = 0. \]

So let us choose \( \lambda \) sufficiently large that (15) holds for \( L^* \) as well as for \( L \). Now

\[ 0 = ((1 + \Delta)^{t-m}w, Lu + \lambda u)_0 \]

\[ = (L^*(1 + \Delta)^{t-m}w + \lambda(1 + \Delta)^{t-m}w, u)_0 \]

for all \( u \in H_t \) and these are dense in \( H_0 \). So

\[ L^*(1 + \Delta)^{t-m}w + \lambda(1 + \Delta)^{t-m}w = 0 \]

and hence (by (15)) \( (1 + \Delta)^{t-m}w = 0 \) so \( w = 0 \). We have proved
Theorem

For every $t$ and for $\lambda$ large enough (depending on $t$) the operator $L + \lambda I$ maps $H_t$ bijectively onto $H_{t-m}$ and $(L + \lambda I)^{-1}$ is bounded independently of $\lambda$. 
As an immediate application we get the important

**Theorem**

*If* $u$ *is a distribution and* $Lu \in H_s$ *then* $u \in H_{s+m}$.

**Proof.**

Write $f = Lu$. By Schwartz’s theorem, we know that $u \in H_k$ for some $k$. So $f + \lambda u \in H_{\min(k,s)}$ for any $\lambda$. Choosing $\lambda$ large enough, we conclude that

$$u = (L + \lambda I)^{-1}(f + \lambda u) \in H_{\min(k+m,s+m)}.$$ 

If $k + m < s + m$ we can repeat the argument to conclude that $u \in H_{\min(k+2m,s+m)}$. We can keep going until we conclude that $u \in H_{s+m}$. 

Shlomo Sternberg

The key idea in this argument goes back to a 1940 paper by Hermann Weyl in the *Annals of Mathematics* entitled “The method of orthogonal projection in potential theory.”
Hermann Klaus Hugo Weyl
(1885-1955)
Notice as an important corollary that any solution of the homogeneous equation $Lu = 0$ is $C^\infty$. Replacing the operator $L$ by $L - \lambda I$ we conclude that any solution of $Lu = \lambda u$ is $C^\infty$.

So if we have found an eigenvector of $L$, we know automatically that it is $C^\infty$. 
We have proved

**Theorem**

*For every* $t$ *and for* $\lambda$ *large enough (depending on* $t$ *) the operator* $L + \lambda I$ *maps* $\mathcal{H}_t$ *bijectively onto* $\mathcal{H}_{t-m}$ *and* $(L + \lambda I)^{-1}$ *is bounded independently of* $\lambda$.

We now draw a second important consequence of this theorem:
Choose $\lambda$ so large that the operators

$$(L + \lambda I)^{-1} \quad \text{and} \quad (L^* + \lambda I)^{-1}$$

exist as bounded operators from $H_0 \rightarrow H_m$. Follow these operators with the injection $\iota_m : H_m \rightarrow H_0$ and set

$$M := \iota_m \circ (L + \lambda I)^{-1}, \quad M^* := \iota_m \circ (L^* + \lambda I)^{-1}.$$ 

Since $\iota_m$ is compact (Rellich’s lemma) and the composite of a compact operator with a bounded operator is compact, we conclude

**Theorem**

*The operators $M$ and $M^*$ are compact.*
Suppose that $L = L^*$. (This is usually expressed by saying that $L$ is “formally self-adjoint”. More on this terminology will come later.) This implies that $M = M^*$. In other words, $M$ is a compact self-adjoint operator, and we can apply the spectral theorem for compact operators to conclude that eigenvectors of $M$ form a basis of $R(M)$ and that the corresponding eigenvalues tend to zero. Our theorem says that $R(M)$ is the same as $υ_m(H_m)$ which is dense in $H_0 = L_2(\mathbb{T})$. We conclude that the eigenvectors of $M$ form a basis of $L_2(\mathbb{T})$. If $Mu = ru$ then $u = (L + \lambda I)Mu = rLu + \lambda ru$ so $u$ is an eigenvector of $L$ with eigenvalue

$$
\frac{1 - r\lambda}{r}.
$$

We conclude that the eigenvectors of $L$ are a basis of $H_0$. 

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\[ \frac{1 - r\lambda}{r}. \]

We conclude that the eigenvectors of \( L \) are a basis of \( H_0 \). We claim that only finitely many of these eigenvalues of \( L \) can be negative. Indeed, since we know that the eigenvalues \( r_n \) of \( M \) approach zero, the numerator in the above expression is positive, for large enough \( n \), and hence if there were infinitely many negative eigenvalues \( \mu_k \), they would have to correspond to negative \( r_k \) and so these \( \mu_k \to -\infty \). I claim that this is impossible:
\[ \|u\|_t \leq c(t)\|Lu + \lambda u\|_{t-m} \tag{15} \]

Indeed, taking \( s_k = -\mu_k \) as the \( \lambda \) in (15) we conclude that \( u = 0 \), if \( Lu = \mu_k u \) if \( k \) is large enough, contradicting the definition of an eigenvector. So all but a finite number of the \( r_n \) are positive, and these tend to zero. To summarize:
Theorem

The eigenvectors of $L$ are $C^\infty$ functions which form a basis of $H_0$. Only finitely many of the eigenvalues $\mu_k$ of $L$ are negative and $\mu_n \to \infty$ as $n \to \infty$. 
It is easy to extend the results obtained above for the torus in two directions. One is to consider functions defined in a domain \( \mathcal{G} \) = bounded open set \( \mathbb{R}^n \) and the other is to consider functions defined on a compact manifold. In both cases a few elementary tricks allow us to reduce to the torus case. We sketch what is involved for the manifold case.
Let $E \to X$ be a vector bundle over a manifold. We assume that $X$ is equipped with a density which we shall denote by $|dx|$ and that $E$ is equipped with a positive definite (smoothly varying) scalar product, so that we can define the $L_2$ norm of a smooth section $s$ of $E$ of compact support:

$$\|s\|_0^2 := \int_M |s|^2(x)|dx|.$$

Suppose for the rest of this lecture that $X$ is compact.
Let \( \{U_i\} \) be a finite cover of \( X \) by coordinate neighborhoods over which \( E \) has a given trivialization, and \( \rho_i \) a partition of unity subordinate to this cover. Let \( \phi_i \) be a diffeomorphism of \( U_i \) with an open subset of \( \mathbb{T}^n \) where \( n \) is the dimension of \( X \). Then if \( s \) is a smooth section of \( E \), we can think of \( (\rho_i s) \circ \phi_i^{-1} \) as an \( \mathbb{R}^m \) or \( \mathbb{C}^m \) valued function on \( \mathbb{T}^n \), and consider the sum of the \( \| \cdot \|_k \) norms applied to each component. We shall continue to denote this sum by \( \| \rho_i f \circ \phi_i^{-1} \|_k \) and then define

\[
\| f \|_k := \sum_i \| \rho_i f \circ \phi_i^{-1} \|_k
\]

where the norms on the right are in the norms on the torus. These norms depend on the trivializations and on the partitions of unity.
These norms depend on the trivializations and on the partitions of unity. But any two norms are equivalent, and the $\| \cdot \|_0$ norm is equivalent to the “intrinsic” $L_2$ norm defined above. We define the Sobolev spaces $W_k$ to be the completion of the space of smooth sections of $E$ relative to the norm $\| \cdot \|_k$ for $k \geq 0$, and these spaces are well defined as topological vector spaces independently of the choices. Since Sobolev’s lemma holds locally, it goes through unchanged.
Similarly Rellich’s lemma: if \( s_n \) is a sequence of elements of \( W_\ell \) which is bounded in the \( \| \|_\ell \) norm for \( \ell > k \), then each of the elements \( \rho_i s_n \circ \phi_i^{-1} \) belong to \( H_\ell \) on the torus, and are bounded in the \( \| \|_\ell \) norm, hence we can select a subsequence of \( \rho_1 s_n \circ \phi_1^{-1} \) which converges in \( H_k \), then a subsubsequence such that \( \rho_i s_n \circ \phi_i^{-1} \) for \( i = 1, 2 \) converge etc. arriving at a subsequence of \( s_n \) which converges in \( W_k \).
A differential operator $L$ mapping sections of $E$ into sections of $E$ is an operator whose local expression (in terms of a trivialization and a coordinate chart) has the form

$$Ls = \sum_{|p| \leq m} \alpha_p(x)D^p s$$

Here the $a_p$ are linear maps (or matrices if our trivializations are in terms of $\mathbb{R}^m$).

Under changes of coordinates and trivializations the change in the coefficients are rather complicated, but the **symbol** of the differential operator

$$\sigma(L)(\xi) := \sum_{|p| = m} a_p(x)\xi^p \quad \xi \in T^*X_x$$

is well defined.
If we put a Riemann metric on the manifold, we can talk about the length $|\xi|$ of any cotangent vector.

If $L$ is a differential operator from $E$ to itself (i.e. $F=E$) we shall call $L$ even elliptic if $m$ is even and there exists some constant $C$ such that

$$\langle v, \sigma(L)(\xi)v \rangle \geq C|\xi|^m|v|^2$$

for all $x \in X$, $v \in E_x$, $\xi \in T^*X_x$ and $\langle , \rangle$ denotes the scalar product on $E_x$. Gårding’s inequality holds. Indeed, locally, this is just a restatement of the (vector valued version) of Gårding’s inequality that we have already proved for the torus. But Stage 4 in the proof extends unchanged (other than the replacement of scalar valued functions by vector valued functions) to the more general case.
We assume knowledge of the basic facts about differentiable manifolds, in particular the existence of an operator $d : \Omega^k \to \Omega^{k+1}$ with its usual properties, where $\Omega^k$ denotes the space of exterior $k$-forms. Also, if $X$ is orientable and carries a Riemann metric then the Riemann metric induces a scalar product on the exterior powers of $T^*X$ and also picks out a volume form. So there is an induced scalar product $(\ , \ ) \equiv (\ , \ )_k$ on $\Omega^k$ and a formal adjoint $\delta$ of $d$

$$\delta : \Omega^k \to \Omega^{k-1}$$

which satisfies

$$(d\psi, \phi) = (\phi, \delta\phi)$$

where $\phi$ is a $(k + 1)$-form and $\psi$ is a $k$-form.
Example: Hodge theory.

The Hodge operator.

Then

$$\Delta := d\delta + \delta d$$

is a second order differential operator on $\Omega^k$ and satisfies

$$(\Delta \phi, \phi) = \|d\phi\|^2 + \|\delta \phi\|^2$$

where $\|\phi\|^2 = (\phi, \phi)$ is the intrinsic $L_2$ norm (so $\|\| = \|\|_0$ in terms of the notation of the preceding section).
Furthermore, if

$$\phi = \sum_{I} \phi_I dx^I$$

is a local expression for the differential form $\phi$, where

$$dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k} \quad I = (i_1, \ldots, i_k)$$

then a local expression for $\Delta$ is

$$\Delta \phi = - \sum g^{ij} \frac{\partial \phi_I}{\partial x^i} \frac{\partial \phi_I}{\partial x^j} + \cdots$$

where

$$g^{ij} = \langle dx^i, dx^j \rangle$$

and the $\cdots$ are lower order derivatives. In particular $\Delta$ is elliptic.
Let $\phi \in \Omega^k$ and suppose that

$$d\phi = 0.$$ 

Let $C(\phi)$, the **cohomology class** of $\phi$ be the set of all $\psi \in \Omega^k$ which satisfy

$$\phi - \psi = d\alpha, \quad \alpha \in \Omega^{k-1}$$

and let

$$\overline{C(\phi)}$$

denote the closure of $C$ in the $L_2$ norm. It is a closed subspace of the Hilbert space obtained by completing $\Omega^k$ relative to its $L_2$ norm. Let us denote this space by $L_2^k$, so $\overline{C(\phi)}$ is a closed subspace of $L_2^k$. 
Theorem

If $\phi \in \Omega^k$ and $d\phi = 0$, there exists a unique $\tau \in C(\phi)$ such that

$$\|\tau\| \leq \|\psi\| \quad \forall \, \psi \in C(\phi).$$

Furthermore, $\tau$ is smooth, and

$$d\tau = 0 \quad and \quad \delta\tau = 0.$$
Proof.

If we choose a minimizing sequence for $||\psi||$ in $\mathcal{C}(\phi)$ we know it is Cauchy, cf. the proof of the existence of orthogonal projections in a Hilbert space. So we know that $\tau$ exists and is unique. For any $\alpha \in \Omega^{k+1}$ we have

$$(\tau, \delta \alpha) = \lim (\psi, \delta \alpha) = \lim (d\psi, \alpha) = 0$$

as $\psi$ ranges over a minimizing sequence. The equation $(\tau, \delta \alpha) = 0$ for all $\alpha \in \Omega^{k+1}$ says that $\tau$ is a weak solution of the equation $d\tau = 0$. 
Proof.

If we choose a minimizing sequence for $\|\psi\|$ in $C(\phi)$ we know it is Cauchy, cf. the proof of the existence of orthogonal projections in a Hilbert space. So we know that $\tau$ exists and is unique. For any $\alpha \in \Omega^{k+1}$ we have

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as $\psi$ ranges over a minimizing sequence. The equation $(\tau, \delta \alpha) = 0$ for all $\alpha \in \Omega^{k+1}$ says that $\tau$ is a weak solution of the equation $d \tau = 0$.

We claim that

$$(\tau, d \beta) = 0 \quad \forall \beta \in \Omega^{k-1}$$

which says that $\tau$ is a weak solution of $\delta \tau = 0$. 

Example: Hodge theory.
Indeed, for any $t \in \mathbb{R}$, 

$$
\|\tau\|^2 \leq \|\tau + td\beta\|^2 = \|\tau\|^2 + t^2\|d\beta\|^2 + 2t(\tau, d\beta)
$$

so 

$$
-2t(\tau, d\beta) \leq t^2\|d\beta\|^2.
$$

If $(\tau, d\beta) \neq 0$, we can choose 

$$
t = -\epsilon \frac{\langle \tau, d\beta \rangle}{|\langle \tau, d\beta \rangle|}, \quad \epsilon > 0
$$

so 

$$
|\langle \tau, d\beta \rangle| \leq \frac{1}{2}\epsilon|d\beta|^2.
$$

As $\epsilon$ is arbitrary, this implies that $(\tau, d\beta) = 0$. 
Example: Hodge theory.

So $(\tau, \Delta \psi) = (\tau, [d\delta + \delta d] \psi) = 0$ for any $\psi \in \Omega^k$. Hence $\tau$ is a weak solution of $\Delta \tau = 0$ and so is smooth. The space $\mathcal{H}^k$ of weak, and hence smooth solutions of $\Delta \tau = 0$ is finite dimensional by the general theory. It is called the space of harmonic forms. We have seen that there is a unique harmonic form in the cohomology class of any closed form, and that the cohomology groups are finite dimensional. □
Example: Hodge theory.

In fact, the general theory tells us that

\[ L^k_2 = \bigoplus_{\lambda} E^k_{\lambda} \]

(Hilbert space direct sum) where \( E^k_{\lambda} \) is the eigenspace with eigenvalue \( \lambda \) of \( \Delta \). Each \( E_{\lambda} \) is finite dimensional and consists of smooth forms, and the \( \lambda \to \infty \). The eigenspace \( E^k_0 \) is just \( \mathcal{H}^k \), the space of harmonic forms. Also, since

\[ (\Delta \phi, \phi) = \|d\phi\|^2 + \|\delta \phi\|^2 \]

we know that all the eigenvalues \( \lambda \) are non-negative.
Since \( d\Delta = d(d\delta + \delta d) = d\delta d = \Delta d \), we see that

\[
d : E^k_\lambda \rightarrow E^{k+1}_\lambda
\]

and similarly

\[
\delta : E^k_\lambda \rightarrow E^{k-1}_\lambda.
\]

For \( \lambda \neq 0 \), if \( \phi \in E^k_\lambda \) and \( d\phi = 0 \), then \( \lambda \phi = \Delta \phi = d\delta \phi \) so \( \phi = d(1/\lambda)\delta \phi \) so \( d \) restricted to the \( E^k_\lambda \) is exact, and similarly so is \( \delta \). Furthermore, on \( \bigoplus_k E^k_\lambda \) we have \( \lambda I = \Delta = (d + \delta)^2 \) so

\[
E^k_\lambda = dE^{k-1}_\lambda \oplus \delta E^{k+1}_\lambda.
\]

This decomposition is orthogonal since \( (d\alpha, \delta\beta) = (d^2\alpha, \beta) = 0 \).
As a first consequence we see that

\[ L^k_2 = \mathcal{H}^k \oplus d\Omega^{k-1} \oplus \delta\Omega^{k+1} \]

(the Hodge decomposition). If \( H \) denotes projection onto the first component, then \( \Delta \) is invertible on the image of \( I - H \) with an inverse there which is compact. So if we let \( N \) denote this inverse on \( \text{im} \ I - H \) and set \( N = 0 \) on \( \mathcal{H}^k \) we get:
Example: Hodge theory.

\[ \Delta N = I - H \]
\[ Nd = dN \]
\[ \delta N = N\delta \]
\[ \Delta N = N\Delta \]
\[ NH = 0 \]

which are the fundamental assertions of Hodge theory, together with the assertion proved above that \( H\phi \) is the unique minimizing element in its cohomology class.
We have seen that

\[ d + \delta : \bigoplus_k E^{2k}_\lambda \rightarrow \bigoplus_k E^{2k+1}_\lambda \]

is an isomorphism for \( \lambda \neq 0 \) (16)

which of course implies that

\[ \sum_k (-1)^k \dim E^k_\lambda = 0 \]

This shows that the index of the operator \( d + \delta \) acting on \( \bigoplus L^k_2 \) is the Euler characteristic of the manifold. (The index of any operator is the difference between the dimensions of the kernel and cokernel).
Example: Hodge theory.

Let $P_{k,\lambda}$ denote the projection of $L^k_2$ onto $E^k_\lambda$. So

$$e^{-t\Delta} = \sum e^{-\lambda t} P_{k,\lambda}$$

is the solution of the heat equation on $L^k_2$. As $t \to \infty$ this approaches the operator $H$ projecting $L^k_2$ onto $\mathcal{H}_k$. Letting $\Delta_k$ denote the operator $\Delta$ on $L^k_2$ we see that

$$\text{tr} e^{-t\Delta_k} = \sum e^{-\lambda_k}$$

where the sum is over all eigenvalues $\lambda_k$ of $\Delta_k$ counted with multiplicity. It follows from (16) that the alternating sum over $k$ of the corresponding sum over non-zero eigenvalues vanishes. Hence

$$\sum (-1)^k \text{tr} e^{-t\Delta_k} = \chi(X)$$

is independent of $t$. The index theorem computes this trace for small values of $t$ in terms of local geometric invariants.
The operator $d + \delta$ is an example of a Dirac operator whose general definition we will not give here. The corresponding assertion and local evaluation is the content of the celebrated Atiyah-Singer index theorem, one of the most important theorems discovered in the twentieth century.
In order to connect what we have done here notation that will come later, it is convenient to let $A = -L$ so that now the operator

$$(zI - A)^{-1}$$

is compact as an operator on $H_0$ for $z$ sufficiently negative. (I have dropped the $\nu_m$ which should come in front of this expression.)
The operator $A$ now has only finitely many positive eigenvalues, with the corresponding spaces of eigenvectors being finite dimensional. In fact, the eigenvectors $\lambda_n = \lambda_n(A)$ (counted with multiplicity) approach $-\infty$ as $n \to \infty$ and the operator $(zI - A)^{-1}$ exists and is a bounded (in fact compact) operator so long as $z \neq \lambda_n$ for any $n$. Indeed, we can write any $u \in H_0$ as

$$u = \sum_{n} a_n \phi_n$$

where $\phi_n$ is an eigenvector of $A$ with eigenvalue $\lambda_n$ and the $\phi$ form an orthonormal basis of $H_0$. 
Then

\[(zI - A)^{-1}u = \sum \frac{1}{z - \lambda_n} a_n \phi_n.\]

The operator \((zI - A)^{-1}\) is called the resolvent of \(A\) at the point \(z\) and denoted by

\[R(z, A)\]

or simply by \(R(z)\) if \(A\) is fixed. So

\[R(z, A) := (zI - A)^{-1}\]

for those values of \(z \in \mathbb{C}\) for which the right hand side is defined.
If \( z \) and \( a \) are complex numbers with \( \text{Re} z > \text{Re} a \), then the integral
\[
\int_0^\infty e^{-zt} e^{at} dt
\]
converges, and we can evaluate it as
\[
\frac{1}{z - a} = \int_0^\infty e^{-zt} e^{at} dt.
\]

If \( \text{Re} z \) is greater than the largest of the eigenvalues of \( A \) we can write
\[
R(z, A) = \int_0^\infty e^{-zt} e^{tA} dt
\]
where we may interpret this equation as a shorthand for doing the integral for the coefficient of each eigenvector, or as an operator valued integral. We will spend a lot of time in this course generalizing this formula and deriving many consequences from it.
Summary,

1. **Review of Sobolev spaces.**
   - Distributions and Schwartz’s theorem.
2. **Gårding’s inequality.**
   - Differential operators.
   - Rellich’s lemma
   - Some numerical inequalities.
   - Elliptic operators.
   - Statement and proof of Gårding’s inequality.
3. **Consequences of Gårding’s inequality.**
4. **Extension of the basic lemmas to manifolds.**
   - Example: Hodge theory.
5. **The resolvent.**