Weak convergence

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November 18, 2010

1. Weak convergence and weak-∗ convergence

Let $X$ be a normed space and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of elements in $X$. Recall that we say that the sequence converges to $x \in X$,

$$x_n \rightarrow x$$

as $n \rightarrow \infty$ if

$$\|x - x_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (1)$$

There is a different notion of convergence which has an important place in analysis.

Definition 1.1. We say that sequence $\{x_n\}_{n \in \mathbb{N}}$ converges weakly to $x \in X$, and we write

$$x_n \underset{w}{\rightarrow} x$$

if, for all $\phi \in X^*$, we have

$$\phi(x_n) \rightarrow \phi(x), \quad \text{as } n \rightarrow \infty.$$

To keep these two notations separate, the term strong convergence is often used to refer to the familiar convergence in norm, (1). When we write $x_n \rightarrow x$ without qualification, we mean strong convergence.

There is a nearly symmetrical notion, which reverses the role of $X$ and $X^*$: this is weak-∗ convergence (pronounced “weak-star”).

Definition 1.2. We say that sequence $\{\phi_n\}_{n \in \mathbb{N}}$ weak-∗ converges to $\phi \in X^*$, and we write

$$\phi_n \underset{w^*}{\rightarrow} \phi$$
if, for all \( x \in X \), we have
\[
\phi_n(x) \longrightarrow \phi(x), \quad \text{as } n \to \infty.
\]

The symmetry between these two definitions is complete only when \( X \) is reflexive. In the reflexive case, consider a sequence of elements \( \phi_n \) in \( X^* \). We can ask whether they are weak-\(*\) convergent to some \( \phi \), as above. We can also ask whether they are weakly convergent to \( \phi \). The second notion involves the elements \( \xi \) in the dual space of \( X^* \), namely \( X^{**} \). In the reflexive case, every such \( \xi \) has the form \( \xi(\phi) = \phi(x) \) for some \( x \in X \). So we then have:
\[
\phi_n \xrightarrow{w} \phi \quad \iff \quad \xi(\phi_n) \longrightarrow \xi(\phi), \quad \forall \xi \in X^{**}
\]
\[
\phi_n(x) \longrightarrow \phi(x), \quad \forall x \in X
\]
\[
\phi_n \xrightarrow{w^*} \phi.
\]

So in a reflexive Banach space (regarded as the dual of its dual), weak-\(*\) convergence and weak convergence are the same thing. (The argument also shows that, whether or not the space is reflexive, a weakly convergent sequence in \( X^* \) is also weak-\(*\) convergent.)

2. Examples

We need to bring the definitions to life with some very simple examples. Our first two examples are from reflexive Banach spaces, where weak and weak-\(*\) convergence are essentially the same thing.

**Hilbert spaces.** Suppose \( X \) is a Hilbert space. By the Riesz representation theorem, every bounded linear functional on \( X \) has the form
\[
x \mapsto (x, a)
\]
for some \( a \in X \). So \( x_n \xrightarrow{w} x \) means
\[
(x_n, a) \to (x, a)
\]
for all \( a \). For example, if \( e_n, n \in \mathbb{N} \), is a complete orthonormal system, then
\[
(e_n, a) \to 0
\]
for all \( a \), because the series \( \sum |(e_n, a)|^2 \) is summable (by Bessel's inequality). So \( e_n \xrightarrow{w} 0 \). It is certainly not the case that \( e_n \to 0 \) however.
The Lebesgue spaces. Similar to the Hilbert space case is the space of $L^p(\mathbb{R})$ with $1 < p < \infty$, whose dual space we can identify with $L^q$ in the standard way, with $(1/p) + (1/q) = 1$. A sequence of functions $b_n$ in $L^p$ converges weakly to $b$ if and only if

$$\int b_n a \rightarrow \int ba$$

for all $a$ in $L^q$.

An example of weak-* but not weak convergence. In the sequence space $l^1$, regarded as the dual space of $c_0$, let $e_n$ be the element with a 1 in the $n$'th spot and 0 elsewhere:

$$e_n = (0, \ldots, 0, 1, 0, \ldots).$$

Then $e_n \xrightarrow{w^*} 0$ as a sequence of elements of $c_0^*$, because for any $a = (a_m) \in c_0$, we have

$$e_n(a) = a_n \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$ 

However it is not true that $e_n \xrightarrow{w} 0$, because the dual space (which we may identify with $l^\infty$) contains the element

$$b = (1, 1, \ldots)$$

and $b(e_n) = 1$ for all $n$. In fact (and this is not very hard to prove), the sequence space $l^1$ has the rather unusual property that every weakly convergent sequence in strongly convergent.

3. Criteria for weak-* convergence

The following result is an application of the Uniform Boundedness Theorem (the Banach-Steinhaus theorem).

**Proposition 3.1.** If $\{\phi_n\}_{n\in\mathbb{N}}$ is a sequence in $X^*$, the dual of a Banach space $X$, and if $\phi_n \xrightarrow{w^*} \phi$, then the sequence is bounded in norm: there exists $M \geq 0$ such that

$$\|\phi_n\| \leq M$$

for all $n \in \mathbb{N}$. 
3. Criteria for weak-∗ convergence

Proof. The deduction of this statement from the Banach–Steinhaus theorem is more or less immediate. If the sequence \( \phi_n(x) \) is convergent to \( \phi(x) \) for all \( x \) (which is what weak-∗ convergence means), then the sequence \( \phi_n(x) \) is bounded. So, for all \( x \), there exists \( M_x \) such that, for all \( n \in \mathbb{N} \),

\[
|\phi_n(x)| \leq M_x.
\]

If we apply the Banach–Steinhaus theorem to the collection of operators \( \{\phi_n : n \in \mathbb{N}\} \subset L(X, F) \), then we learn that there is an \( M \) such that \( \|\phi_n\| \leq M \) for all \( n \).

The next proposition is a sort of converse to the previous one.

**Proposition 3.2.** Let \( X \) be a separable normed space. Let \( \{\phi_n\}_{n \in \mathbb{N}} \) be a sequence in \( X^* \) that is bounded in norm:

\[
|\phi_n| \leq M.
\]

Then there is a subsequence \( \{\phi_{n_j}\}_{j \in \mathbb{N}} \) that is weak-∗ convergent:

\[
\phi_{n_j} \overset{w^*}{\longrightarrow} \phi
\]

as \( j \to \infty \).

The following lemma is half of what is going on in this proposition. The lemma is useful in its own right.

**Lemma 3.3.** Let \( X \) be a normed space and \( Q \) a dense subset of \( X \). Let \( \{\phi_n\} \) be a sequence in \( X^* \), with

\[
\|\phi_n\| \leq M, \quad \text{for all } n \in \mathbb{N}.
\]

Suppose that, for all \( q \in Q \), the sequence of real or complex numbers \( \{\phi_n(q)\} \) is convergent. Then the sequence \( \{\phi_n(x)\} \) is convergent for all \( x \) in \( X \), and the limit \( \phi(x) \) defines a bounded linear functional on \( X \).

Remark. The conclusion of the lemma means in particular that \( \phi_n \) weak-∗ converges to \( \phi \). The Lemma also holds also for a sequence of operators \( T_n \in L(X, Y) \) for a Banach space \( Y \), rather than just in \( X^* = L(X, F) \). The proof for the more general case is exactly the same.

**Proof of the lemma.** The proof uses a standard sort of 3\( \epsilon \)-argument. Given \( x \) in \( X \), we will show that the sequence \( \{\phi_n(x)\} \) is Cauchy. To do this, choose \( q \in Q \) with \( \|x - q\| \leq \epsilon / (3M) \), as we can do because \( Q \) is dense. We then have

\[
|\phi_n(x) - \phi_n(q)| \leq \epsilon / 3
\]
for all \( n \). Since \( \{ \phi_n(q) \} \) is Cauchy, we can choose \( n_0 \) such that
\[
|\phi_n(q) - \phi_m(q)| \leq \epsilon/3
\]
for all \( n, m \geq n_0 \). The triangle inequality then gives us the Cauchy property for \( \phi_n(x) \):
\[
|\phi_n(x) - \phi_m(x)| \leq \epsilon
\]
for \( n, m \geq n_0 \). So there exists for each \( x \) an element \( \phi(x) \) in \( \mathbb{R} \) or \( \mathbb{C} \) which is the limit of the Cauchy sequence:
\[
\phi_n(x) \to \phi(x).
\]
It is easy to verify that the limit \( \phi \) is linear, and is also bounded, with \( \|\phi\| \leq M \).

**Proof of Proposition 3.2.** Let \( Q \) be a countable dense subset of the separable normed space \( X \). Write
\[
Q = \{ q_1, q_2, \ldots \}.
\]
For each \( q \in Q \), the sequence \( \phi_n(q) \) is a bounded sequence of real or complex numbers. So there is a subsequence indexed by \( \mathbb{N}_q \subset \mathbb{N} \) such that these numbers converge:
\[
\lim_{n \in \mathbb{N}_q, n \to \infty} \phi_n(q) \to \phi(q).
\]
We now use a “diagonal subsequence” argument. We can suppose that
\[
\mathbb{N}_{q_1} \supset \mathbb{N}_{q_2} \supset \cdots,
\]
and we then construct a new subsequence \( \phi_{n_j} \) indexed by \( j \in \mathbb{N} \) by taking \( n_j \) to be the \( j \)'th element of \( \mathbb{N}_{q_j} \). Then
\[
\phi_{n_j}(q) \to \phi(q)
\]
for all \( q \) in \( Q \). The lemma above now applies, and tells us that \( \phi \) extends to a bounded linear functional on all of \( X \) and \( \phi_n(x) \to \phi(x) \) for all \( x \). In other words,
\[
\phi_n \overset{w^*}{\longrightarrow} \phi.
\]

**Corollary 3.4.** If \( \{ x_n \} \) is a bounded sequence in a separable, reflexive Banach space \( X \), then there is a weakly convergent subsequence.
Proof. We have $X \cong X^{**}$ via the canonical isomorphism $x \mapsto \hat{x}$. So $X^{**}$ is separable, because $X$ is. If the dual of a Banach space is separable, then so is the original Banach space; so the separability of $X^{**}$ means that $X^*$ is separable too.

The sequence $\{\hat{x}_n\}$ is thus a bounded sequence in the dual of a separable Banach space $X^*$. The proposition tells us it has a weak-∗ convergent subsequence. But weak-∗ convergence for $\hat{x}_n$ is the same thing as weak convergence for $x_n$. □

As a particular example, a sequence of functions in $L^p(\mathbb{R})$ ($1 < p < \infty$) whose $L^p$ norms are bounded always has a weakly convergent subsequence.

Exercise. Use the material of this section to prove the following criterion for weak convergence in $L^p(\mathbb{R})$, $1 < p < \infty$. A sequence of functions $\{f_n\}$ in $L^p$ is weakly convergent if and only if it satisfies both of the following two conditions:

(a) the $L^p$ norms are bounded: there exists $M$ such that $\|f_n\|_p \leq M$ for all $n$; and

(b) for every $a < b$, the sequence of integrals $\int_a^b f_n$ is convergent.

4. Further properties of weak convergence

Here are two further properties of weak and weak-∗ convergence. The proof of both of these are part of the problem sets, so we omit them here.

**Proposition 4.1.** Let $\phi_n \in X^*$ and $\phi_n \overset{w^*}{\longrightarrow} \phi$ as $n \to \infty$, then

$$\|\phi\| \leq \liminf_{n \to \infty} \|\phi_n\|.$$ 

The proof is not hard: just start with the definition of $\|\phi\|$. As an illustration, recall that an orthonormal sequence in a Hilbert space converges weakly (and therefore weak-∗) to zero; so in this case, the norm of the limit is zero, but the lim inf of the norms of the terms is 1.

The next proposition is a little more subtle. It concerns the case when we have equality in the above inequality. It does not hold in all Banach spaces: we state it for a Hilbert space.
Proposition 4.2. Let \( \phi_n \) be a sequence in a Hilbert space \( H \). Suppose that \( \phi_n \xrightarrow{w} \phi \) and that
\[
\|\phi\| = \lim_{n \to \infty} \|\phi_n\|.
\]
Then we actually have strong convergence:
\[
\phi_n \to \phi
\]
in \( H \).

In this case of a Hilbert space, the proof of the Proposition is rather straightforward. (Hint. You are trying to prove that \( \langle \phi - \phi_n, \phi - \phi_n \rangle \) converges to zero.) The proposition does not hold in all normed spaces. For example, in the sequence space \( c_0 \), let \( e_n \) be the usual element with a 1 in the \( n \)'th spot. Then we have
\[
e_1 + e_n \xrightarrow{w} e_1
\]
as \( n \to \infty \), and also
\[
\|e_1\| = \lim_{n \to \infty} \|e_1 + e_n\| = 1.
\]
But \( e_1 + e_n \) is not converging strongly to \( e_1 \).