A Detailed Price Discrimination Example

Suppose that there are two different types of customers for a monopolist’s product. Customers of type 1 have demand curves as follows. These demand curves include subscripts for prices to allow for the possibility that the monopolist can charge different prices for each type of customer:

\[ Q_1(p_1) = 150 - p_1 \]
\[ Q_2(p_2) = 200 - 4p_2 \]

For simplicity, we assume throughout that the monopolist has no fixed costs and we consider separately the monopolist’s profit maximizing strategies with two different forms of marginal cost – constant marginal cost with \( MC(Q) = c \) and increasing marginal cost with \( MC(Q) = kQ^2 \).

**Case 1: Third-Degree Price Discrimination**

If the monopolist can set different prices for each type of consumer, the profit maximizing quantity for each group will be based on its marginal revenue. To compute marginal revenue, first reverse the relationship of price and quantity:

\[ p_1(Q_1) = 150 - Q_1 \]
\[ p_2(Q_2) = 50 - Q_2 / 4 \]

Next multiply price by quantity to find total revenues and differentiate by quantity to find marginal revenue for each type of consumer.

\[ TR_1(Q_1) = (150 - Q_1) Q_1; \quad MR_1(Q_1) = 150 - 2Q_1 \]
\[ TR_2(Q_2) = (50 - Q_2 / 4) Q_2; \quad MR_2(Q_2) = 50 - Q_2 / 2 \]

**Case 1a: Constant MC = c**

With constant marginal cost, the monopolist can simply solve separate profit maximization problems to determine distinct prices and quantities for each group. The two problems can be solved independently because the price and quantity for one group has no effect on the revenue or costs for production and sales to the other group.

Marginal revenue is strictly decreasing in quantity for each group, so the monopolist’s profit maximizing quantity for type 1 occurs when \( MR_1 = MC = c \). Specifically, the optimal choice of \( Q_1 \) is given by \( MR_1 = 150 - 2Q_1 = c \), with solution \( Q_1^* = 75 - c / 2 \). Similarly the optimal choice of \( Q_2 \) is given by \( MR_2 = 50 - Q_2 / 2 = c \), or \( Q_2^* = 100 - 2c \).

For example, if \( c = 0 \), then the profit maximizing solutions and revenue maximizing solutions coincide: \( Q_1^* = 75, \ p_1^* = 75, \ Q_2^* = 100, \ p_1^* = 25 \). If \( c > 50 \) then the formula for \( Q_2^* \) gives a negative quantity. In this case, the marginal cost of each unit is higher than the price any buyer of type 2 is willing to pay, so the optimal choice is \( Q_2^* = 0 \). Similarly, if \( c > 150 \), then the optimal choice is \( Q_1^* = 0 \).
Figure 1 compares the marginal revenues for the two types of consumers, where the dotted line represents the marginal revenue for type 2’s. Points A and B represent the optimal choices of quantities (75 for type 1 and 100 for type 2) with constant marginal cost of $c = 0$.

As the constant marginal cost $c$ increases, the profit maximizing quantities to sell to each type of consumer decrease. So for example, if $c = 30$, then the optimal quantities are $Q_1^* = 60$, $Q_2^* = 40$. Consistent with our analysis above, the graph indicates that there should be no sales to type 2’s if $c > 50$ and no sales to type 1’s if $c > 150$. 
Case 1b: Increasing Marginal Cost: \( C(Q) = kQ^2 \)

With increasing marginal cost, the quantity sold to one type of consumer affects the marginal cost of production for the other type of consumer. As a result, the first-order conditions for the two profit maximization problems are interdependent and must be solved simultaneously. With quadratic costs, \( C(Q) = kQ^2 \), marginal cost \( MC(Q) = 2kQ \) is linear in (total) quantity produced, so the two first-order conditions produce a linear system of two equations and two unknowns (\( Q_1 \) and \( Q_2 \)).

\[
\begin{align*}
MR_1(Q_1) &= MC(Q_{\text{total}}) \\
150 - 2Q_1 &= 2k (Q_1 + Q_2) \\
\text{OR} \quad 50 - Q_2 / 2 &= 2k (Q_1 + Q_2) \\
MR_2(Q_2) &= MC(Q_{\text{total}}) \\
50 - Q_2 / 2 &= 2k (Q_1 + Q_2) \\
\end{align*}
\]

Solving (1) for \( Q_1 \) gives

\[
Q_1 = \frac{150 - 2kQ_2}{2k + 2}
\]

Substituting for \( Q_1 \) in (2) gives

\[
\begin{align*}
50 - Q_2 / 2 &= 2k Q_2 + 2k (150 - 2kQ_2) / (2k + 2) \\
\text{OR} \quad (50 - Q_2 / 2) (2k + 2) &= 2k (2k + 2) Q_2 + 2k (150 - 2kQ_2) \\
\text{OR} \quad 50 (2k + 2) - Q_2 (k + 1) &= (4k^2 + 4k - 4k^2) Q_2 - 300k \\
\text{OR} \quad 100 - 200k &= Q_2 (5k + 1) \\
\text{OR} \quad Q_2^* &= (100 - 200k) / (5k + 1)
\end{align*}
\]

This equation yields a positive solution for \( Q_2^* \) if \( k < \frac{1}{2} \). This outcome makes sense because \( MR_1(Q_1 = 50) = MR_2(Q_2 = 0) = 50 = MC(Q_{\text{total}} = 50) \) if \( k = \frac{1}{2} \). The monopolist maximizes profits by allocating the first 50 units produced to type 1 (since \( MR_1(Q_1) > MR_2(Q_2 = 0) \) for \( Q_1 < 50 \)). But if \( k \geq 1/2 \), then the monopolist exhausts all gains from production with \( MR_1(Q_1) = MC(Q_1) \) for \( Q_1 < 50 \) – implying \( Q_2^* = 0 \).

So, if \( k \geq 1/2 \), the monopolist can simply solve an ordinary profit maximization problem for type 1 consumers alone. Then the first-order condition \( MR_1(Q_1) = MC(Q_1) \) is \( 150 - 2Q_1 = 2k Q_1 \) and has solution \( Q_1^* = 150 / (2 + 2k), p_1^* = 150 (1 + 2k) / (2 + 2k) \).

Further, if \( k < \frac{1}{2} \), the monopolist will sell positive quantities to each type of consumer. Substituting (4) into (3) gives

\[
Q_1^*(k \leq \frac{1}{2}) = \frac{150 / (2k + 2) - 2k(100 - 200k)}{[(2k + 2) (5k + 1)]} \\
= \frac{[150 (5k + 1) - 2k(100 - 200k)]}{[(2k + 2) (5k + 1)]} \\
= \frac{(150 + 550k + 400k^2)}{[(2k + 2) (5k + 1)]} \\
= \frac{(75 + 200k)}{(5k + 1)}
\]

So for example, if \( k = 0.05 \), then \( Q_1^* = 68 \) and \( Q_2^* = 72 \).
Figure 2 illustrates the comparison of marginal revenues and marginal costs for optimal price discrimination when the marginal cost varies with the total quantity sold. The monopolist sells the first 50 units to type 1’s, since $\text{MR}_1 = \text{MR}_2$ when $Q_1 = 50$, $Q_2 = 0$. On the second part of the monopolist’s marginal revenue curve, the monopolist sells to both types of consumers and sets $\text{MR}_1 = \text{MR}_2$ so that $150 - 2Q_1 = 50 - Q_2 / 2$, which simplifies to $Q_1 = 50 + Q_2 / 4$. This formula indicates that when the monopolist sells to both consumers, it sells four units to type 2’s for each additional unit sold to type 1’s. Total quantity sold is $Q_t = Q_1 + Q_2 = 50 + 5Q_2 / 4$, or $Q_2 = 4Q_t / 5 - 40$. Note that this formula gives $Q_2 > 0$ for $Q_t > 50$, consistent with the conclusion that the monopolist sells the first 50 units to type 1’s.

We can identify the marginal revenue corresponding to total quantity $Q_t$ by computing $\text{MR}_2$ as a function of $Q_t$. That is, the second part of the monopolist’s marginal revenue curve in Figure 2, is given by

$$\text{MR}_r(Q_t) = \text{MR}_2(Q_2) = 50 - Q_2 / 2 = 50 - (4Q_t/5 - 40) / 2 = 70 - 2Q_t/5.$$  

Figure 2 also includes the linear marginal cost curve with $\text{C}(Q_{\text{total}}) = Q_{\text{total}}^2 / 20$ so that $\text{MC}(Q_{\text{total}}) = Q_{\text{total}} / 10$. Then, $\text{MR} = \text{MC} = 14$ at $Q_{\text{total}} = 140$, corresponding to $Q_1^* = 68$ and $Q_2^* = 72$ – the same values calculated above with $\text{C}(Q_{\text{total}}) = kQ_{\text{total}}^2$ and $k = 0.05$. 
Case 2: Uniform Pricing

If it is impossible for the monopolist to distinguish between customers (or illegal to charge different prices for different people buying the same product), then \( p_1 = p_2 = p \). Based on the formulas for \( Q_1 \) and \( Q_2 \), consumers of type 1 will buy a positive quantity of the good for \( p < 150 \), while consumers of type 2 will buy a positive quantity of the good for \( p < 50 \).

Based on this analysis, total demand \( Q_t \) is simply the demand from consumers of type 1 for high prices – \( p \geq 50 \). By contrast, total demand is the sum of the demands for the two different types of consumers for low prices – \( p < 50 \).

\[
Q_t(p) = Q_1(p) = 150 - p \quad \text{if } p \geq 50.
\]
\[
Q_t(p) = Q_1(p) + Q_2(p) = 350 - 5p \quad \text{if } p < 50.
\]

Note that it does not matter whether we use one formula or the other at the transition point, \( p = 50 \), because each formula gives the same value at this point (since \( Q_2(50) = 0 \)).

Because of the change of slope (sometimes called a "kink") in the total demand function at \( p = 50 \), there are several possible optimal outcomes. There are separate first-order conditions on each portion of the demand function, each of which can identify a local maximum. It is also possible to find an optimum in profits at the transition point, though we will see that this is impossible in the context of this monopoly problem.

To proceed with formal analysis, first reverse the relationship between \( p \) and \( Q_t \) on each portion of the demand curve – noting that the transition point between the two portions of the demand curve occurs at \( Q_t(50) = 100 \).

\[
p(Q_t) = 150 - Q_t \quad \text{if } Q_t < 100.
\]
\[
p(Q_t) = 70 - Q_t / 5 \quad \text{if } Q_t > 100
\]

Multiplying quantity by price, then differentiating with respect to \( Q_t \), this means that total revenues and marginal revenues are given by

\[
TR(Q_t) = (150 - Q_t)Q_t \quad \text{if } Q_t \leq 100.
\]
\[
TR(Q_t) = (70 - Q_t / 5)Q_t \quad \text{if } Q_t > 100
\]

\[
MR(Q_t) = 150 - 2Q_t \quad \text{if } Q_t < 100.
\]
\[
MR(Q_t) = 70 - 2Q_t / 5 \quad \text{if } Q_t > 100
\]

Quantity and total revenue are continuous at the transition point \( (p = 50, Q_t = 100) \), but marginal revenue is discontinuous at this point – for \( Q_t \) just less than 100, \( MR = -50 \), but for \( Q_t \) just greater than 100, \( MR = 30 \). This discontinuity arises because of the entrance of type 2 consumers into the market. Quantity \( Q_t = 100 \) corresponds to \( p = 50 \). If \( p > 50 \), (i.e. \( Q_t < 100 \)), type 2 consumers do not purchase the good and so the marginal revenue is computed for type 1 consumers alone. So, if \( p < 50 \), (i.e. \( Q_t > 100 \)), marginal revenue \( MR(Q_t) \) incorporates the purchases of both types of consumers.
Returning to the formulas for the individual marginal revenues, \( \text{MR}_1 \) is negative and \( \text{MR}_2 \) is positive at the kink in the demand curve (\( Q_1 = 100, Q_2 = 0, p = 50 \)). Since this point represents the first time that a type 2 consumer purchases the good, \( \text{MR}_2 \) is simply equal to the maximum price paid by a type 2 consumer and must be positive.

We expect a jump in marginal revenues at the kink in the demand curve since \( \text{MR}_2 \) is positive at this point (and type 2 consumers are not included in marginal revenue computations until that point). For this reason, it cannot be optimal for the monopolist to produce at the quantity \( Q_k \) at the kink in demand based on the following two cases:

**Possibility 1:** \( \text{MR} \) jumps at \( Q_k \) but is still negative or zero. Then it must increase profits to reduce quantity from \( Q_k \).

**Possibility 2:** \( \text{MR} \) jumps at \( Q_k \) and is positive. Then it must increase profits to increase quantity from \( Q_k \).

Because of the jump in the marginal revenue, it will often be the case that \( \text{MR} \) jumps from negative to positive at \( Q_k \) (corresponding to a version of possibility 2 above). Then the first-order condition \( \text{MR} = \text{MC} \) could be satisfied on both segments of the demand curve – indicating a local maximum in profits at two different quantities. Then it’s necessary to compare the profits at these two local optimal points to determine which of them is profit maximizing.

Intuitively, the monopolist must absorb losses in sales to consumers of type 1 beyond the optimal level in order to reap the benefits of sales to consumers of type 2. For the specific demand functions in the example, even if \( c = 0 \), the monopolist’s profit maximizing quantity for type 1 consumers is less than 100 – the quantity necessary to bring type 2 consumers into the market. If the gains from selling to type 2’s outweigh the costs of the additional sales to type 1’s, then it would be optimal to sell to both types of consumers and otherwise it is optimal to sell to only type 1’s.

### Case 2a: Constant \( \text{MC} = c \)

The first-order conditions with \( \text{MC} = c \) are given by

\[
\begin{align*}
150 - 2Q_t & = c & \text{if } Q_t < 100 \\
70 - 2Q_t / 5 & = c & \text{if } Q_t > 100
\end{align*}
\]

Equation (6) yields the solution \( Q_{t1}^* = 75 - c / 2 \). Since this formula always satisfies the condition \( Q_t < 100 \), there is a local maximum where \( \text{MR} = \text{MC} \) on the first segment of the demand curve with \( Q_t < 100 \).

Equation (7) yields the solution \( Q_{t2}^* = 175 - 5c / 2 \). This formula yields a solution with \( Q_t > 100 \) if \( c < 30 \), indicating that there is a second local maximum where \( \text{MR} = \text{MC} \) on the second segment of the demand curve with \( Q_t > 100 \).
Combining these observations, if \( MC = c > 30 \), there is a unique local maximum given by \( Q_{t1}^* = 75 - c / 2 \) and the monopolist only sells to type 1 consumers. However, if \( c < 30 \), there are two local maxima – one before and one after the kink in demand at \( Q_{k} \) – then the monopolist must compare their profits to determine which one is preferable.

**Figure 3: Marginal Revenue with Uniform Pricing**

This graph suggests two immediate observations. First, for \( 0 < c < 30 \), we can directly identify the points \( Q_{t1}^* \) and \( Q_{t2}^* \) by finding the quantities on each of the segments of the marginal revenue lines where \( MR = c \). Second, as \( c \) increases, \( Q_{t1}^* \) decreases and the losses incurred by producing units from \( Q_{t1}^* \) to 100 increase. Similarly as \( c \) increases, \( Q_{t2}^* \) decreases and the profits for producing units from 100 to \( Q_{t2}^* \) to 100 decrease as well. That is, as \( c \) increases, the cost to the monopolist of producing a large enough quantity to sell to type 2 consumers increases and the gains from selling to type 2 consumers decreases.

Based on these two observations, we expect that the monopolist will prefer \( Q_{t1}^* \) to \( Q_{t2}^* \) with constant marginal costs close to 30 and that monopolist might prefer \( Q_{t2}^* \) to \( Q_{t1}^* \) with constant marginal costs close to 0. Formally, since \( Q_{t1}^* \) and \( Q_{t2}^* \) are separate solutions to the local first order conditions with constant MC and \( 0 < c < 30 \), we have to compute the separate profits for these two values and compare them to determine which is the optimal choice for the monopolist.
Figure 4 graphs the monopolist’s (uniform pricing) profit as a function of total quantity with \( MC = 0 \) and \( MC = 10 \). The kink in the demand function causes a dramatic change in the shape of the profit function at \( Q = 100 \). In each case, there is a local maximum in profits with \( Q < 100 \) and a separate maximum in profits with \( Q > 100 \).

Consistent with our observations above, the local maxima occur at relatively lower quantities with \( MC = 10 \) (when \( Q_{t1}^* = 70 \) and \( Q_{t2}^* = 150 \)) than with \( MC = 0 \) (when \( Q_{t1}^* = 75 \) and \( Q_{t2}^* = 175 \)). Also consistent with the observations above, the increase in marginal cost makes \( Q_{t1}^* \) more attractive by comparison to \( Q_{t2}^* \). For these particular values of \( MC = c \), the profit maximizing choice is \( Q_{t2}^* = 175 \) when \( c = 0 \) and \( Q_{t1}^* = 70 \) when \( c = 10 \). (Some further algebraic computations indicate that \( Q_{t1}^* \) is profit maximizing with constant marginal cost \( c > c^* = 50 – 8*5^{1/2} \approx 5.27 \), and \( Q_{t2}^* \) is profit maximizing with constant marginal cost \( c < c^* \).)
Case 2b: Increasing Marginal Cost: $C(Q) = kQ^2$

The first-order conditions with $MC = 2kQ$ are given by

\[
egin{align*}
150 - 2Q_t &= 2kQ_t & \text{if } Q_t < 100 \\
70 - 2Q_t / 5 &= 2kQ_t & \text{if } Q_t > 100
\end{align*}
\] (8)

Equation (7) yields the solution $Q_{t1}^* = 150 / (2 + 2k)$. Since this formula always satisfies the condition $Q_t < 100$, there is a local maximum where $MR = MC$ on the first segment of the demand curve with $Q_t < 100$.

Equation (8) yields the solution $Q_{t2}^* = 350 / (2 + 10k)$. This formula yields a solution with $Q_t > 100$ if $k < .15$, indicating that there is a second local maximum where $MR = MC$ on the second segment of the demand curve with $Q_t > 100$.

Combining these observations, if $C(Q) = kQ^2$ and $k > .015$, there is a unique local maximum given by $Q_{t1}^* = 150 / (2 + 2k)$ and the monopolist only sells to type 1 consumers. However, if $k < .15$, there are two local maxima – one before and one after the kink in demand at $Q_k$, and the monopolist must compare the profits at these two points to determine which one is preferable.

As in Case 2(a), we expect to find that an increase in marginal cost – i.e. an increase in $k$ – tends to make $Q_{t1}^*$ relatively more attractive by comparison to $Q_{t2}^*$. Some trial and error calculations support this conclusion – if $k > k^* \approx 0.023$, then the monopolist maximizes profits by only selling to type 1 consumers with $Q_{t1}^* = 150 / (2 + 2k)$, but if instead $k < k^*$, the monopolist maximizes profits by selling to both types of consumers with $Q_{t2}^* = 350 / (2 + 10k)$.