Banach spaces: definitions and examples

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1. Normed spaces and Banach spaces

Let $F$ denote either the field $\mathbb{R}$ or $\mathbb{C}$. A normed vector space is a vector space $X$ over $F$ together with a function (the norm),

$$\| \cdot \| : X \to \mathbb{R}^\geq$$

satisfying

- $\| x \| = 0$ if and only if $x = 0$;
- $\| \lambda x \| = |\lambda| \| x \|$ for all $x \in X$ and $\lambda \in F$;
- $\| x + y \| \leq \| x \| + \| y \|$.

These conditions mean in particular that we can define a distance function $d(x, y) = \| x - y \|$ and obtain a metric space $(X, d)$. So in a normed space, we can talk about all the things that go with metric spaces: open and closed sets, Cauchy sequences and convergence, and so on. In particular, a set $S \subset X$ is open if for all $x \in X$ there exists a $\delta > 0$ such that the ball $B(x; \delta)$ is entirely contained in $S$. Here

$$B(x; \delta) = \{ x \mid \| x - y \| \leq \delta \}.$$

And a sequence $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence if for all $\epsilon > 0$, there exists $n(\epsilon)$ such that for all $m, n \geq n(\epsilon)$ we have

$$\| x_m - x_n \| \leq \epsilon.$$
As with any metric space, we say that a normed space is complete if every Cauchy sequence \((x_n)_{n \in \mathbb{N}}\) converges: that is, there exists \(x \in X\) with

\[
\|x_n - x\| \to 0, \quad \text{as } n \to \infty.
\]

We write \(x_n \to x\) as \(n \to \infty\).

**Definition 1.1.** A Banach space (over \(\mathbb{R}\) or \(\mathbb{C}\)) is a complete normed vector space.

If \(X\) is a vector space with two different norms, \(\| \cdot \|\) and \(\| \cdot \|^\prime\), then the norms are said to be equivalent if there are positive constants \(C_1\) and \(C_2\) such that

\[
C_1\|x\| \leq \|x\|^\prime \leq C_2\|x\|
\]

for all \(x \in X\). If two norms are equivalent, then Cauchy sequences for one norm are Cauchy sequences for the other. The same goes for convergent sequences. So one of the norms makes \(X\) complete if and only if the other does.

**2. A criterion for completeness**

The following criterion for completeness of a normed space is often useful. (We used it first in proving that \(L^1\) is complete.) If \(X\) is a normed space and \(v_n, n \in \mathbb{N}\), are elements of \(X\), then we say that the series is absolutely summable if the series

\[
\sum_n \|v_n\|
\]

is convergent. We say that the series is convergent if the sequence of partial sums converges in \(X\): that is, if \(s_n = \sum_{m \leq n} v_m\), then we ask that there exists \(s \in X\) with \(s_n \to s\) as \(n \to \infty\). Then we have the following:

**Lemma 2.1.** A normed space \(X\) is complete if and only if every absolutely summable series is convergent.

**Proof.** We do the “if” direction (which is the more interesting one). Suppose every absolutely summable series in \(X\) converges. Let \(x_n, n \in \mathbb{N}\) be a Cauchy sequence. To show that \(X\) is complete, we must show that the sequence \(x_n\) converges. For each \(j \geq 1\), we can find \(n_j\) such that

\[
\|x_n - x_{n'}\| \leq 2^{-j}
\]
for all \( n, n' \geq n_j \). Define

\[ v_1 = x_{n_1} \]

and define

\[ v_j = x_{n_j} - x_{n_{j-1}} \]

for \( j \geq 2 \). These definitions achieve the following. We have

\[ \|v_j\| \leq 2^{-j} \]

so that the series \( \sum v_j \) is absolutely summable, and hence convergent, by our hypothesis. We also have arranged that the partial sum

\[ s_j = \sum_{i<j} v_i \]

is equal to \( x_{n_j} \). Because the series \( \sum v_j \) is convergent, we have

\[ s_j \to s, \]

as \( j \to \infty \). In other words, \( x_{n_j} \to s \) as \( j \to \infty \). This is saying that a subsequence of the original Cauchy sequence \( \{x_n\}_{n \in \mathbb{N}} \) converges. But it follows easily that the original sequence converges. Indeed, given \( \epsilon > 0 \), we can find \( n_* \) so that

\[ \|x_n - x_{n'}\| \leq \epsilon/2 \]

for all \( n, n' \geq n_* \); and we can find \( j \) so that \( n_j \geq n_* \)

\[ \|x_{n_j} - s\| \leq \epsilon/2. \]

From these and the triangle inequality, it follows that \( \|x_n - s\| \leq \epsilon \) for all \( n \geq n_* \). \( \square \)

### 3. Some examples

We give some examples of Banach spaces. In some of these examples, there are things we need to verify to check that we have a Banach space: the triangle inequality for the norm, and completeness. We omit these proofs. For the interesting case of \( L^p(\mathbb{R}^d) \), we will give proofs in a separate handout. We again use \( F \) to denote

**The space** \( c_0 \). This is the space of all sequence \( a = (a_n)_{n \in \mathbb{N}} \) with \( a_n \in F \) such that \( a_n \to 0 \) as \( n \to \infty \). The norm is

\[ \|a\|_\infty = \sup_n |a_n|. \]
The space $\ell^\infty$. This is the space of all bounded sequences $a = (a_n)_{n \in \mathbb{N}}$ with $a_n \in F$. The norm is 
\[ \|a\| = \sup_n |a_n|. \]

The space $\ell^p$. For $1 \leq p < \infty$, this is the space of all sequences $a = (a_n)_{n \in \mathbb{N}}$, with $a_n \in \mathbb{R}$, such that
\[ \sum_{n=1}^{\infty} |a_n|^p \]
is convergent. The norm is
\[ \|a\|_p = \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p}. \]

The triangle inequality for $\ell^p$ is a version of the Minkowski inequality.

All these sequence spaces can be considered with sequences indexed by any countable set in place of $\mathbb{N}$. For example, we will write $\ell^p(\mathbb{Z})$ to denote the space of sequences $(a_n)$ indexed by $n \in \mathbb{Z}$ such that $\sum_{n=\infty}^{-\infty} |a_n|^p$ is finite.

The space $C(\Omega)$. If $\Omega$ is a compact metric space, $C(\Omega)$ is the space of continuous functions on $\Omega$ with values in $F$. The norm is the sup norm:
\[ \|f\|_0 = \sup_{\omega \in \Omega} |f(\omega)|. \]

The space $C_0(\mathbb{R}^d)$. This is the space of continuous functions $f : \mathbb{R}^d \to F$ satisfying $|f(x)| \to 0$ as $|x| \to \infty$. The norm is the sup norm, $\sup_{x \in \mathbb{R}^d} |f(x)|$.

The space $L^p(\mathbb{R}^d)$. For $1 < p < \infty$, the Lebesgue space $L^p(\mathbb{R}^d)$ is the space of equivalence classes of measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $|f|^p$ is integrable. The norm is
\[ \|f\|_p = \left( \int |f|^p \right)^{1/p}. \]
The space $L^\infty(\mathbb{R}^d)$. A measurable function $f$ is essentially bounded if there exists $M$ such that $|f| \leq M$ almost everywhere. The space of such essentially bounded functions is a Banach space, with the norm being

$$\|f\|_\infty = \inf\{ M : |f| \leq M \text{ a.e. } \}.$$ 

The space $\mathbb{F}$. The field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is itself a Banach space, with norm equal to the absolute value or complex modulus $|x|$.

The space $\mathbb{F}^n$. The space $\mathbb{R}^n$ or $\mathbb{C}^n$ is a Banach space, and can be given a variety of norms. For example, the norms from the sequence spaces $\ell^p$ for $1 \leq p \leq \infty$ all give rise to norms of a similar sort on $\mathbb{F}^n$. Any two norms on a finite-dimensional vector space are equivalent. (This is outlined in a problem set.) A norm on any vector space is completely determined by describing the set of vectors of length less than or equal to 1: the closed unit ball about the origin. In the case of $\mathbb{R}^n$, the unit ball for any norm has to be a compact, convex subset of $\mathbb{R}^n$ which is invariant under the symmetry $x \mapsto -x$. Any such convex set corresponds to a norm.

4. Bounded linear operators.

Let $X$ and $Y$ be normed spaces. A linear map $T : X \to Y$ is bounded if there exists an $M \geq 0$ such that

$$\|Tx\|_Y \leq M\|x\|_X$$

for all $x \in X$. The smallest such $M$ is the norm or operator norm of $T$, and is written $\|T\|$. As long as $X$ is non-zero, the norm $\|T\|$ can be described variously as

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$$

or as

$$\|T\| = \sup_{\|x\|=1} \|Tx\|.$$ 

We write $L(X, Y)$ for the set of all bounded linear operators from $X$ to $Y$.

Lemma 4.1. Equipped with the operator norm, $L(X, Y)$ is a normed linear space.
4. Bounded linear operators.

Proof. We must check that if \( T_1 \) and \( T_2 \) are bounded linear operators, then so are \( \lambda T_1 \) and \( T_1 + T_2 \). This is straightforward, and the verification shows

\[
\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|
\]

and

\[
\|\lambda T_1\| = |\lambda| \|T_1\|.
\]

For example, for the first of these, we just use

\[
\|(T_1 + T_2)x\| = \|T_1x + T_2x\| \\
\leq \|T_1x\| + \|T_2x\| \\
\leq \|T_1\|\|x\| + \|T_2\|\|x\| \\
= (\|T_1\| + \|T_2\|)\|x\|.
\]

Finally we must note that \( \|T\| = 0 \) only if \( T = 0 \), which is clear from the definition. \( \square \)

So far, we have not assumed that either \( X \) or \( Y \) is complete. We now look at completeness:

Lemma 4.2. If \( Y \) is complete, then so is the normed space \( L(X, Y) \) with the operator norm. That is, \( L(X, Y) \) is a Banach space.

Proof. Suppose \( Y \) is complete (a Banach space). Let \( T_n, n \in \mathbb{N} \), be a Cauchy sequence in \( L(X, Y) \). This means that for all \( \epsilon > 0 \) we can find \( n(\epsilon) \) such that

\[
\|T_n - T_m\| \leq \epsilon
\]

for all \( n, m \geq n(\epsilon) \). We will show that \( T_n \) converges to some \( T \in L(X, Y) \) in operator norm.

For each fixed \( x \in X \), we can consider the sequence \( T_nx \). This is a Cauchy sequence in \( Y \), because

\[
\|T_nx - T_mx\| \leq \epsilon\|x\|
\]

for all \( n, m \geq n(\epsilon) \). So for each \( x \in X \) there exists a unique \( y \) in \( Y \) such that

\[
T_nx \to y.
\]

Define \( T : X \to Y \) by \( Tx = y \). We omit the easy verification that \( T \) is a linear map. We must check that \( T \) is bounded. To do this, we take the limit as \( n \to \infty \) in (1) to get

\[
\|Tx - T_mx\| \leq \epsilon\|x\|
\]
for \( m \geq n(\epsilon) \), and then we use the triangle inequality to deduce

\[
\|Tx\| \leq \|T_m\|\|x\| + \epsilon\|x\|.
\]

This shows that \( T \) is bounded, with \( \|T\| \leq \|T_m\| + \epsilon \), for any \( m \geq n(\epsilon) \). The inequality (2) also shows that \( \|T - T_m\| \leq \epsilon \) for \( m \geq n(\epsilon) \), so \( T_m \to T \) in operator norm as \( m \to \infty \).

If there are mutually inverse bounded linear maps \( T : X \to Y \) and \( S : Y \to X \), then the Banach spaces \( X \) and \( Y \) are equivalent. If \( T \) (and hence \( S \) also) preserves the norm, so that \( \|x\| = \|Tx\| \) for all \( x \), then \( X \) and \( Y \) are isometrically isomorphic.

5. Dual spaces

If \( X \) is a normed space over \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{C} \), then we can consider the space \( L(X, \mathbb{F}) \). By the previous lemma, this is a Banach space. It is called the dual space of \( X \). We write it \( X^* \). Here are some examples, without proof. Some proofs will appear in another handout.

- The dual space of \( c_0 \) is (isometrically isomorphic to) \( \ell^1 \). The isomorphism is given as follows. For \( a = (a_n) \) in \( \ell^1 \), we define an element \( \alpha \in (c_0)^* \) by
  \[
  \alpha(b) = \sum_n a_n b_n.
  \]

- The dual space of \( \ell^1 \) is (isometrically isomorphic to) \( \ell^\infty \).

- The dual space of \( \ell^p \) for \( 1 < p < \infty \) is isometrically isomorphic to \( \ell^q \), where \( q \) satisfies \( 1/p + 1/q = 1 \) (the “dual exponent”). In particular \( \ell^2 \) is its own dual.

- The dual space of \( C_0(\mathbb{R}^d) \) is isometrically isomorphic to \( L^1(\mathbb{R}^d) \). The isomorphism is as follows. Given \( a \) in \( L^1(\mathbb{R}^d) \), we define \( \alpha \in C_0(\mathbb{R}^d)^* \) by
  \[
  \alpha(f) = \int_{\mathbb{R}^d} f(f)
  \]
  for \( f \in C_0(\mathbb{R}^d) \).

- The dual space of \( L^1(\mathbb{R}^d) \) is isometrically isomorphic to \( L^\infty(\mathbb{R}^d) \).

- For \( 1 < p < \infty \), the dual space of \( L^p(\mathbb{R}^d) \) is \( L^q(\mathbb{R}^d) \), where \( q \) is the dual exponent.
• The dual space of $L^\infty(\mathbb{R}^d)$ is something rather large and unmanageable. An element $\alpha$ in the dual space of $L^\infty([0,1])$ is completely determined by specifying its value on the $\chi_E$ for all measurable subsets $E$ of $[0,1]$. These values $n_E \in \mathbb{R}$ or $\mathbb{C}$ are subject only to the constraint that they be additive under finite unions of disjoint sets: $n_{E_1} + n_{E_2} = n_{E_1 \cup E_2}$ if $E_1 \cap E_2$ is null.

6. Dense subsets

A subset $S \subset X$ in normed space $X$ is dense if every point of $X$ is a limit point of points of $X$. An infinite-dimensional Banach space may have dense linear subspaces. Here are some examples:

**Proposition 6.1.** In the Lebesgue space $L^p(\mathbb{R}^d)$, for $1 \leq p < \infty$, the following subspaces are dense.

- The functions $f$ of bounded support: those that vanish outside a bounded set $\{|x| \leq R\}$.
- The bounded functions of bounded support.
- The simple functions.
- The step functions.
- The continuous functions of bounded support.
- The functions of Schwartz class.

In proving that subspaces such as these are dense in $L^p$, the following is useful:

**Proposition 6.2 (Dominated convergence theorem, formulated for $L^p$).** Let $f_n$ be a sequence in $L^p(\mathbb{R}^d)$ with $f_n \to f$ almost everywhere. Suppose that there exists $g$ in $L^p(\mathbb{R}^d)$, with $|f_n| \leq g$ almost everywhere for all $n$. Then $f$ is in $L^p$ and

$$\|f - f_n\|_p \to 0$$

as $n \to \infty$.

A Banach space is separable if it contains a countable dense subset. The Lebesgue space $L^p$ is separable for $1 \leq p < \infty$, because a dense subset is the set of step functions

$$g = \sum_{n=1}^{N} a_n \chi_{R_n}$$
where the coefficients $a_n$ are rational numbers and the rectangles $R_n$ have rational coordinates for their vertices. This does not work for $L^\infty$. You will show in the problem sets that $L^\infty$ is not separable.