Bézout’s theorem

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This handout gives a statement and proof of Bézout’s theorem concerning the intersection of two plane curves. The background to the proof is: (i) the definition of intersection multiplicity which we gave in class and which is presented in a separate handout; and (ii) the basic facts about resultants.

The presentation here is rather different from that in Kirwan’s book, because our definition of intersection multiplicity has a different starting point. (Kirwan defines intersection multiplicity using resultants.) The price to be paid is a longer proof, with more abstract algebra. On the other hand, we avoid some awkward questions about the independence of the choice of coordinates that arise when using resultants as Kirwan does. We also deviate from Kirwan’s presentation in that we present Bézout’s theorem in the context of affine plane curves (that is, curves in $\mathbb{C}^2$) rather than curves in the projective plane.

1. Statement of Bézout’s theorem

Let $P$ and $Q$ be non-zero polynomials belonging to $\mathbb{C}[x, y]$. An earlier handout gave a definition of the intersection multiplicity, $I_p(P, Q)$, for any $p \in \mathbb{C}^2$. In the case that $P$ and $Q$ each have no repeated factors, we recast this as the intersection multiplicity at $p$ of the two algebraic curves, $C$ and $D$ in $\mathbb{C}^2$, defined by the polynomials $P$ and $Q$ respectively; we may then write it as $I_p(C, D)$.

Bézout’s theorem is concerned with the number of intersection points of two such curves, counted with multiplicity: that is

$$\sum_{p \in C \cap D} I_p(C, D).$$

The theorem states if $C$ and $D$ do not have a common component then this sum is at most $nm$, where $n$ and $m$ are the degrees of the defining polynomials $P$ and $Q$ for the curves. Further, the theorem provides a criterion for when equality occurs.

Before going further, let us explain the criterion which will ensure equality. Without assuming that $P$ and $Q$ have no repeated factors, let us continue to write $n$ and $m$ for their degrees. Let $P_s$ be the homogeneous polynomial of degree $n$ obtained by assembling all the monomials from $P$ whose degree in
(x, y) is exactly n. Similarly, let \( Q_* \) be the homogeneous part of \( Q \) of degree \( m \). Recall that a homogeneous polynomial in two variables (over \( \mathbb{C} \)) will always factor as a product of linear factors. This means that we can write

\[
P_* = \prod_{i=1}^{n} (\alpha_i x + \beta_i y) \\
Q_* = \prod_{j=1}^{m} (\gamma_j x + \delta_j y).
\]

The criterion for equality in Bézout’s theorem is that \( P_* \) and \( Q_* \) have no common factor: that is, one requires that none of the linear factors \( (\alpha_i x + \beta_i y) \) of \( P_* \) is a scalar multiple of one of the factors \( (\gamma_j x + \delta_j y) \) of \( Q_* \). Here then is the statement of Bézout’s theorem:

**Theorem 1.** Let \( P \) and \( Q \) be polynomials in \( \mathbb{C}[x, y] \) of degrees \( n \) and \( m \). Suppose that \( P \) and \( Q \) have no common factor. Then

\[
\sum_{p \in \mathbb{C}^2} I_p(P, Q) \leq nm.
\]

If \( P_* \) and \( Q_* \) are the homogeneous parts of \( P \) and \( Q \) of degree \( n \) and \( m \) respectively, and if \( P_* \) and \( Q_* \) have no common factor, then we have equality:

\[
\sum_{p \in \mathbb{C}^2} I_p(P, Q) = nm.
\]

Some remarks are in order here. First, as suggested in the statement of the theorem, the condition that \( P_* \) and \( Q_* \) have no common factor implies that \( P \) and \( Q \) have no common factor (a simple exercise). Second, the intersection multiplicity \( I_p(P, Q) \) is zero unless \( p \) lies on the common zero set of the two polynomials, so the sum could be written

\[
\sum_{p \in \mathbb{C} \cap D} I_p(P, Q)
\]

where \( C \) and \( D \) are the curves that \( P \) and \( Q \) define. As we saw in class, \( C \cap D \) is a finite set when \( P \) and \( Q \) have no common factor, so the sum that appears in Bézout’s theorem has only finitely many non-zero terms.

**2. The case of two lines, or a line and a curve**

To start with the simplest case, let us suppose that \( P \) and \( Q \) each have degree 1. So

\[
P = \alpha x + \beta y + r \\
Q = \gamma x + \delta y + s
\]
for constants $r$ and $s$. The homogeneous parts of top degree are

$$
P_s = \alpha x + \beta y$$
$$Q_s = \gamma x + \delta y.$$

The zero sets of $P$ and $Q$ are lines $M$ and $N$ in $\mathbb{C}^2$. The statement that $P$ and $Q$ have no common factor is the statement that $P$ is not a multiple of $Q$ in this case, or equivalently that the lines $M$ and $N$ are distinct. The polynomials $P_s$ and $Q_s$ define lines $M_s$ and $N_s$ which pass through $(0, 0)$ and which are “parallel” to $M$ and $N$ respectively. Here we have borrowed the word “parallel” from plane geometry in $\mathbb{R}^2$. The condition that $P_s$ and $Q_s$ have no common factor is the condition that the lines $M_s$ and $N_s$ are distinct: we can rephrase this as saying that $M$ and $N$ are not parallel. Algebraically, the condition is that $P_s$ and $Q_s$ are not proportional. Bézout’s theorem simply says that two distinct lines $M$ and $N$ meet in at most one point; and that the if the lines are not parallel then they meet in exactly one point.

3. Breaking up the proof into two steps

Recall that, in the previous handout on intersection multiplicity, we defined $I_p(P, Q)$ as the dimension of a quotient,

$$I_p(P, Q) = \dim \left( \frac{L_p}{\langle P, Q \rangle_p} \right).$$

Here $L_p \subset \mathbb{C}(x, y)$ is the ring of rational functions $a/b$ whose denominator $b$ is non-zero at $p$, and $\langle P, Q \rangle_p$ is our notation for the ideal in $L_p$ generated by the polynomials $P$ and $Q$. When we say “dimension”, we mean here the dimension of a complex vector space. We can rephrase Bézout’s theorem as saying that the dimension of the vector space

$$\bigoplus_{p \in \mathbb{C}^2} \left( \frac{L_p}{\langle P, Q \rangle_p} \right)$$

is at most $mn$, and is equal to $mn$ if $P_s$ and $Q_s$ have no common factor. Once again, although we have written a direct sum of vector spaces indexed by $p$ in $\mathbb{C}^2$, only a finite number of these vector spaces are non-zero, and we could equally have written

$$\bigoplus_{p \in \mathbb{C} \cap D} \left( \frac{L_p}{\langle P, Q \rangle_p} \right).$$

To emphasize the structure of the argument, let us write $L$ for the ring of polynomials, $\mathbb{C}[x, y]$, and let us write $\langle P, Q \rangle$ for the ideal in $L$ generated by $P$
and \(Q\). We can consider the quotient
\[
\frac{L}{\langle P, Q \rangle}.
\]

Our proof of Bézout’s theorem has two steps, summarized by the two propositions below. The first proposition makes no reference to the local rings \(L_p\) and concerns only the ring \(L\) and its quotient:

**Proposition 1.** If \(P\) and \(Q\) are of degree \(n\) and \(m\) and have no common factor, then the dimension of the vector space
\[
\frac{L}{\langle P, Q \rangle}
\]
is at most \(mn\). Furthermore, if \(P_\ast\) and \(Q_\ast\) have no common factor, then the dimension is exactly \(mn\).

The second proposition is the following:

**Proposition 2.** If \(P\) and \(Q\) have no common factor, then there is an isomorphism
\[
\frac{L}{\langle P, Q \rangle} \cong \bigoplus_{p \in \mathbb{C} \cap D} \left( \frac{L_p}{\langle P, Q \rangle_p} \right).
\]

Bézout’s theorem follows from these two propositions, for (as we observed just above), the vector space that appears on the right of the isomorphism in the second proposition has total dimension equal to the sum of the intersection multiplicities, \(I_p(P, Q)\), by definition of \(I_p(P, Q)\).

## 4. Proof of Proposition 1

For any \(d \geq 0\), let us write \(L(d)\) for the linear space of polynomials \(G \in L\) whose degree in the \(x\) variable is at most \(d\). Such polynomials can be written
\[
G = g_0(y) + g_1(y)x + \cdots + g_d(y)x^d
\]
where the degrees of the polynomials \(g_i(y)\) are arbitrary. In this way, we can think of \(L(d)\) (as a vector space) as:
\[
L(d) \cong \mathbb{C}[y] \oplus \cdots \oplus \mathbb{C}[y] = \mathbb{C}[y]^{d+1}. \tag{1}
\]

We will say that a polynomial \(G\) as above is *monic in the \(x\) variable* if the leading coefficient \(g_d(y)\) is 1.
Hypothesis. We will assume henceforth that the polynomial $P$, which has degree $n$ in $(x, y)$, also has degree exactly $n$ in the $x$ variable. This can be arranged by a linear change of coordinates: it is equivalent to saying that $P_x$ is not divisible by $y$. After replacing $P$ by a scalar multiple, we may as well assume also that $P$ is monic in the $x$ variable.

Lemma 1. Under the above hypothesis on $P$, the linear map

$$\pi : L(n + m - 1) \to L / \langle P, Q \rangle$$

$$\pi(S) = S + \langle P, Q \rangle$$

is surjective, and its kernel consists of all polynomials which can be expressed in the form

$$GP + HQ$$

with $G \in L(m - 1)$ and $H \in L(n - 1)$.

Proof. To show surjectivity, consider a typical element $T + \langle P, Q \rangle$ in the quotient $L / \langle P, Q \rangle$. Because $P$ is monic in $x$, the usual division algorithm works, and we can “divide $T$ by $P$ with remainder”: that is, we can write $T$ as

$$T = S + KP$$

for some polynomials $S$ and $K$, with the degree of $S$ in the $x$ variable being less than $n$. Thus $S$ belongs to $L(n - 1)$ and certainly therefore to $L(n + m - 1)$ also. Because $KP$ belongs to the ideal, we have

$$S + \langle P, Q \rangle = T + \langle P, Q \rangle.$$ 

This means that $\pi(S) = T + \langle P, Q \rangle$, and we have succeeded in showing that $T + \langle P, Q \rangle$ is in the image of $\pi$.

Suppose now that $S$ is in the kernel of $\pi$. This means that $S$ has degree $n + m - 1$ or less in $x$, and

$$S + \langle P, Q \rangle = 0 + \langle P, Q \rangle,$$

or in other words, $S$ belongs to the ideal. We can therefore write

$$S = GP + HQ$$

for some $G$ and $H$ in $L$. Using the division algorithm again, we can “divide $H$ by $P$ with remainder $H_1$”; thus,

$$H = H_1 + AP$$

for $H_1$ in $L(n - 1)$. Define

$$G_1 = G - AQ.$$
We then have
\[ S = G_1 P + H_1 Q. \]
Because \( H_1 \) is in \( L(n - 1) \), the degree in \( x \) of \( H_1 Q \) is at most \( n + m_1 \). Since \( P \) is degree exactly \( n \), it follows that \( G_1 \) has degree at most \( m_1 \) in \( x \), for otherwise the degree of the expression on the right would be bigger than \( n + m_1 \) while \( S \) on the left has degree \( n + m - 1 \) or less. Thus we have expressed the typical element \( S \) in the kernel as \( G_1 P + H_1 Q \), with \( G_1 \) and \( H_1 \) in \( L(m - 1) \) and \( L(n - 1) \) as desired. Conversely, it is clear that any element of this form is in the kernel of \( \pi \).

To continue the proof of Proposition 1, we now see from the lemma above (and the first isomorphism theorem) that the space \( L/(P, Q) \) whose dimension we seek is isomorphic to another space,
\[ \frac{L}{(P, Q)} \cong \frac{L(n + m - 1)}{L(m - 1)P + L(n - 1)Q} \]
(2)
where the notation on the denominator on the right is short for the set of all \( GP + HQ \) with \( G \) and \( H \) in \( L(m - 1) \) and \( L(n - 1) \) respectively. We can write this as
\[ L(n + m - 1)/\text{Im}(R) \]
where \( R \) is the map
\[ R : L(n + m - 1) = L(m - 1) \times L(n - 1) \rightarrow L(n + m - 1) \]
given by
\[ (G, H) \mapsto GP + HQ. \]
If we identify \( L(d) \) with \( \mathbb{C}[d]^{N+1} \) as in (1), then this becomes a map
\[ R : \mathbb{C}[y]^{m+n} \rightarrow \mathbb{C}[y]^{m+n} \]
given a matrix \( R \) whose entries are polynomials in \( y \): specifically \( R \) is the matrix that is familiar in the construction of the resultant of \( P \) and \( Q \):
\[
R = \begin{bmatrix}
p_0 & 0 & \cdots & 0 & q_0 & 0 & \cdots & 0 \\
p_1 & p_0 & \cdots & 0 & q_1 & q_0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \\
p_n & p_{n-1} & p_0 & & 0 & & & \\
0 & p_n & p_1 & q_0 & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & \\
0 & 0 & \cdots & p_n & 0 & 0 & \cdots & q_m
\end{bmatrix}
\]
(3)
Here each \( p_i \) for example is a polynomial in \( y \), and
\[ P(x, y) = p_0(y) + \cdots + p_n(y)x^n. \]
The polynomials \( q_j \) come from \( Q \) similarly; but there is a slight difference, because we have not assumed that the degree of \( Q \) in the \( x \) variable is equal to its total degree – we therefore allow that.

The next lemma concerns such square matrices of polynomials in general:

**Lemma 2.** Let \( R \) be any \( N \times N \) matrix whose entries \( r_{ij} \) are polynomials in \( y \) with complex coefficients. Let \( R(y) \) be the determinant of \( R \) (a polynomial in \( y \)), and suppose that \( R(y) \) is not the zero polynomial. Then the dimension of the complex vector space \( \mathbb{C}[y]^N / \text{Im}(R) \) is equal to the degree of \( R(y) \).

**Proof.** This touches on some standard material from Math 123. Look first at the case \( N = 1 \). In this case, we are looking at \( \mathbb{C}[y]/(r) \), where \((r)\) is the ideal consisting of all multiples of a fixed polynomial \( r(x) \in \mathbb{C}[y] \). The dimension of this quotient is equal to the degree of \( r \), because (using the division algorithm) every coset \( s + (r) \) has a unique coset representative \( \tilde{s} \) of degree less than or equal to \( \deg(r) - 1 \).

For the case of general \( N \), consider first the case that \( R \) is a diagonal matrix. This case follows from the case \( N = 1 \), for \( \mathbb{C}[y]^N / \text{Im}(R) \) is just a direct sum of \( N \) vector spaces, each of the form \( \mathbb{C}[y]/(r_{ii}) \). Finally, for a general matrix \( R \), we use the fact that \( R \) can be reduced to diagonal form by “row and column operations”. More particularly, the operations we need are: interchange two rows (or columns); multiply a row (or column) by a non-zero scalar; add a \( s(y) \) times one row (or column) to another, for an arbitrary polynomials \( s(y) \). These row and column operations leave the vector space \( \mathbb{C}[y]^N / \text{Im}(R) \) unchanged (up to isomorphism) and change \( R = \det R \) only by scalar multiples. \( \square \)

In our particular case, the determinant of the matrix \( R \) at (3) is the resultant \( R_{P,Q}(y) \). This was almost our definition of the resultant, except that \( m \) is not necessarily the degree of \( Q \) in the \( x \) variable, so \( q_m \) may be zero. We saw in class that this slight discrepancy only changes the determinant by some power of \( p_n(y) \); and we have assumed that \( p_n = 1 \), so all is well. So, from (2), we obtain

\[
\dim \frac{L}{\langle P, Q \rangle} = \deg R_{P,Q}(y).
\]

So the next lemma completes the proof of Proposition 1.

**Lemma 3.** Let \( P \) and \( Q \) have degrees \( n \) and \( m \), and let \( P_*, Q_* \) be as before. Suppose further that \( P \) satisfies our hypothesis, that its degree in \( x \) is equal to its total degree. Then the resultant \( R_{P,Q}(y) \) as degree at most \( nm \) in \( y \). Furthermore, if \( P_* \) and \( Q_* \) have no common factor, then the degree of the resultant is exactly \( nm \).

**Proof.** The degree of \( p_i(y) \) is at most \( n - i \), and the degree of \( q_j(y) \) is at most \( m - j \). If we write \( r_{ij} \) for the \( ij \) entry of \( R \) above, then

\[
\deg r_{ij}(y) \leq \begin{cases} n - i + j, & j \leq m \\ -i + j, & j > m. \end{cases}
\]
The typical term in the determinant is a product
\[ \prod_{j=1}^{n+m} r_{\sigma(j),j} \]
for some permutation \( \sigma \). The degree of such a term is bounded by
\[ nm + \sum_{j=1}^{n+m} (-\sigma(j) + j) \]
which is just \( nm \).

This argument also shows that the coefficient of \( y^{nm} \) in the resultant only depends on the terms of \( P_\infty \) and \( Q_\infty \) (the terms of top degree). So to prove the “furthermore” part of the lemma, it is enough to show that the resultant of \( P_\infty \) and \( Q_\infty \) has degree exactly \( nm \) if \( P_\infty \) and \( Q_\infty \) share no common linear factor. To do this, we can use the multiplicative property of the resultant, namely the fact that \( R_{P,Q_1,Q_2} \) is \( R_{P,Q_1}R_{P,Q_2} \). Each of \( P_\infty \) and \( Q_\infty \) is a product of linear factors (\( n \) and \( m \) of them, respectively). So we need only check that the resultant of \( (\alpha x + \beta y) \) and \( (\gamma x + \delta y) \) is a non-zero multiple of \( y \), provided that \( \alpha \) is non-zero and the two linear polynomials are independent. This is straightforward: the resultant is \( (\beta \gamma - \alpha \delta)y \).

5. Proof of Proposition 2

To complete the proof of Bézout’s theorem, we now turn to the second of our two proposition: Proposition 2 above. It is no longer necessary to assume that the degree of \( P \) in the \( x \) variable is equal to its total degree.

To prove Proposition 2, we will define a linear map
\[ \Phi : L \to \bigoplus_{p \in C \cap D} \left( \frac{L_p}{(P, Q)_p} \right) \]
and we will show:

(a) the map \( \Phi \) is surjective;
(b) the kernel of the map \( \Phi \) is the ideal \( (P, Q) \).

The result then follows from the first isomorphism theorem. The definition of \( \Phi \) is as follows. A polynomial \( S \in L \) can be regarded as an element of the local ring \( L_p \), for each \( p \in C \cap D \), because of the inclusion \( L \subset L_p \). We can therefore define a map
\[ \phi_p : L \to \frac{L_p}{(P, Q)_p} \]
The map $\Phi$ is defined as the direct sum of these:

$$
\Phi = \bigoplus_{p \in C \cap D} \phi_p
$$

$$
\Phi(S) = (\phi_{p_1}(S), \ldots, \phi_{p_N}(S)).
$$

(Here $p_1, \ldots, p_N$ is some enumeration of the finite set $C \cap D$.)

Let us show that $\Phi$ is surjective, as claimed in (a) above. Suppose we are given an element

$$
\bar{f}_p = f_p + \langle P, Q \rangle_p
$$

in $L_p/\langle P, Q \rangle_p$, for each $p$. We seek an $S$ with

$$
\Phi(S) = (\bar{f}_{p_1}, \ldots, \bar{f}_{p_N}).
$$

Concretely, this equality means that for each $p \in C \cap D$ we have

$$
S \equiv f_p \pmod{\langle P, Q \rangle_p}.
$$

In the previous handout on intersection multiplicity, we saw that there is a $k \geq 0$ such that

$$
\langle (x - x(p))^k, (y - y(p))^k \rangle_p \subset \langle P, Q \rangle_p,
$$

where $(x(p), y(p))$ are the coordinates of $p$. So to achieve (4), it is sufficient if we find $S$ with

$$
S \equiv f_p \pmod{\langle (x - x(p))^k, (y - y(p))^k \rangle_p}
$$

for all $p$. The ideal that appears here is the ideal of rational which vanish at $p$ along with their all their partial derivatives $\partial^i+j f/\partial x^i \partial y^j$ for $i, j < k$. In other words, we are seeking a polynomial $S$ whose derivatives of these orders at the point $p$ agree with the derivatives of $f_p$, for each $p$ in the finite set $C \cap D$. It is a quite elementary exercise to find a polynomial whose derivatives of various orders are specified at some finite set of points, and this observation completes the proof that $\Phi$ is surjective.

We now turn to showing that the kernel of $\Phi$ is $\langle P, Q \rangle$, as claimed in (b). First of all, it is clear that $P$ and $Q$ both belong to the kernel, so the kernel does contain the ideal they generate. We need to show the reverse inclusion. So let $S$ be an element of the kernel of $\Phi$. This means that, for each $p$ in $C \cap D$, we have $S \in \langle P, Q \rangle_p$. Concretely, this means that there are rational functions $f_p$ and $g_p$ which are regular at $p$ and such that

$$
S = f_p P + g_p Q.
$$

(5)
We wish to show that $S \in \langle P, Q \rangle$. To this end, define

$$ I = \{ T \in L | TS \in \langle P, Q \rangle \}. $$

From the definition, it is easy to verify that $I$ is an ideal of the ring $L$. If we can show that $1 \in I$, then we will know that $S$ belongs to $\langle P, Q \rangle$ and we will be done.

**Lemma 4.** For each $p \in \mathbb{C}^2$, there is a polynomial $T$ in the ideal $I$ with $T(p) \neq 0$.

**Proof.** First consider the case that $p \in C \cap D$. We know that there are rational functions $f_p$ and $g_p$ in the local ring at $p$ such that (5) holds. Write $f_p$ and $g_p$ as the ratios of polynomials, say $f_p = a_p/b_p$ and $g_p = c_p/d_p$, so that $b_p$ and $d_p$ are non-zero at $p$. Clearing denominators, we get

$$(b_pd_p)S = a_pd_pP + c_pb_pQ.$$  

This means that the polynomial $T = b_pd_p$ belongs to the ideal $I$. Since $T$ is non-zero at $p$, we are done proving the lemma in the case $p \in C \cap D$.

If $p$ does not belong to $C \cap D$, then one of $P$ or $Q$ is non-zero at $p$. We may suppose $P$ is non-zero at $p$. The definition of the ideal $I$ shows that $P$ belongs to $I$. Since $P(p)$ is non-zero, we can just take $T = P$ and we are done.

Since our remaining task is to show that $1$ belongs to $I$, the following lemma now completes the proof.

**Lemma 5.** Let $I$ be an ideal in the ring $L = \mathbb{C}[x, y]$. Suppose that, for each $p \in \mathbb{C}^2$, there is an element $T$ in $L$ with $T(p) \neq 0$. Then $1$ belongs to $I$ (from which it follows, of course, that $I = L$).

**Proof.** This is Hilbert’s Nullstellensatz, in a different guise from the one we saw it in. Here is an ad hoc argument for this case, leveraging what we already know about curves in $\mathbb{C}^2$. First, let $G$ be any non-zero element of $I$ and let $C$ be the curve it defines. Some elementary linear algebra allows us to see that, for any finite set of points $p_1, \ldots, p_N$ in $\mathbb{C}^2$, there is an element $H$ in the ideal $I$ which vanishes at none of the $p_i$. We pick one point in each irreducible component of the curve $C$ and apply this observation to obtain a polynomial $H_1$ whose zero-set $D_1$ has no component in common with $C$. The intersection $C \cap D_1$ is therefore finite. Using the same idea again, we construct another element $H_2$ whose zero set $D_2$ has the property that $C \cap D_2$ is finite and disjoint from $C \cap D_1$. We can suppose coordinates are chosen so that the $y$-coordinates of the points of $C \cap D_1$ are different from the $y$-coordinates of the points of $C \cap D_2$.

Let $R_1(y)$ and $R_2(y)$ be the resultants of the pairs $(G, H_1)$ and $(G, H_2)$ (eliminating the $x$ variable). Regarded as elements of $L$, each $R_i$ belongs to the ideal generated by $G$ and $H_i$, and therefore also to the ideal $I$. The zeros of these resultants are the $y$-coordinates of the corresponding intersection points, and we have arranged for these two sets to be disjoint. Thus $R_1$ and $R_2$ are coprime
polynomials in \( \mathbb{C}[y] \), and it follows that there are polynomials \( U(y) \) and \( V(y) \) with \( UR_1 + VR_2 = 1 \). Since we already saw that both the \( R_i \) belonged to the ideal \( I \), it follows that 1 belongs to \( I \), as claimed. \( \square \)

The proof of Proposition 2 is now complete. An examination of the proof shows that we established more than just an isomorphism of vector spaces. The left- and right-hand sides in the proposition are isomorphic as rings, and also as modules over the ring \( L = \mathbb{C}[x, y] \).

Exercise. Analyze the proof to see where we used the fact that the complex numbers are algebraically closed.