1 From the neoclassical model to Tobin’s Q theory of investment

**Neoclassical model:** firms invest up to the point where the marginal productivity of capital is equal to its real rental price:

\[ MPK = \frac{R}{P} \]

and the real rental price of capital \( R/P \) is determined by cost of owning capital:

\[ \frac{R}{P} = \left( i - \frac{\Delta P_k}{P_k} + \delta \right) \left( \frac{P_k}{P} \right) = \frac{P_k}{P} (r + \delta) \]

assuming that the price of capital grows at the general inflation rate. In words, there are three costs to owning capital: (i) you lose the nominal interest rate, which is the yield on an alternative investment; (ii) part of the capital stock depreciates every period; (iii) the price of capital may fall (negative capital gain). Thus net investment is a function of the difference between the marginal product of capital and its cost:

\[ Net\ Investment = I_n \left( MPK - \frac{P_k}{P} (r + \delta) \right) \]

and gross investment is \( I = I_n + \delta K \). This is the idea behind \( I(r) \) functions in models like IS-LM. Since MPK decreases in \( K \), there is investment or disinvestment until in the long-run equilibrium we have

\[ MPK = \frac{P_k}{P} (r + \delta) \quad (steady-state\ condition) \]

What is missing is this model? One issue is that firms don’t invest only if additional capital brings extra profits in the current period; instead, the investment decision is forward-looking.

**Tobin’s Q theory:** the rate of investment is determined by one variable, \( q \), defined as:

\[ q = \frac{\text{Market value of installed capital}}{\text{Replacement cost of installed capital}} \]

The market value of installed capital is set by the stock market valuation of the company. If \( q > 1 \), it means the firm can raise its stock market value by acquiring more capital, so it will invest beyond what is needed to cover depreciation. If \( q < 1 \), capital is more expensive to buy than its valuation by the market, so the firm lets its capital stock depreciate without fully replacing it. In a steady-state equilibrium, \( q = 1 \), attained through diminishing returns to capital (\( q \) is brought down by firms investing more and more, or up by firms shrinking their capital stock).

The main difference with the previous approach is that the \( q \) theory incorporates future expected profits in the calculation, not just current profits. It provides one explanation of a link between stock prices and the real economy: higher stock prices encourage firms to invest.

What is still missing? We have been assuming that firms can costlessly adjust their capital stock when they have an incentive to do so. More plausibly, there are costs to changing the size of the physical capital stock: you can’t jump from low \( K \) to high \( K \) overnight without incurring some adjustment costs (e.g. installing the new machines and training workers to use them, renting some more space, or conversely dismantling the old machines). Also, there might be financial constraints preventing firms from engaging in large and sudden expansions of their capital stock.
2 Q theory with adjustment costs

This is a discrete-time version of the model seen in class, from Romer’s *Advanced macroeconomics*.

Assumptions:

- There are $N$ identical firms in an industry.
- A firm’s profits depend negatively on the aggregate capital stock of the industry $K_t$: if the sector faces a downward-sloping demand curve, the bigger your competitors, the lower your profits.
- A firm’s profits at time $t$ are proportional to its own capital stock (size of the firm) $k_t$. Implicitly, we are assuming constant returns to scale. So each firm makes profits $\pi (K_t) k_t$.
- Adjusting the capital stock costs, on top of the price of the new capital goods, and additional $C(K_{t+1} - K_t)$, such that $C(0) = 0$, $C'(0) = 0$ and $C''(\cdot) > 0$. Adjustment costs are always positive and the marginal adjustment cost increases in the size of the adjustment.
- The interest rate $r$ is constant over time.
- Capital does not depreciate ($\delta = 0$).

The firm’s objective function is

$$\max_{(k_t, I_t)} \sum_{t=0}^{\infty} \left( \frac{1}{1 + r} \right)^t [\pi (K_t) k_t - I_t - C(I_t)]$$

s.t. $k_t = k_{t-1} + I_t$ for all $t$

Let $\lambda_t$ be the Lagrange multiplier for the period $t$ constraint, and $q_t \equiv (1 + r)^t \lambda_t$. Then we can write the maximization problem with a Lagrangian as

$$\max_{(k_t, I_t)} L = \sum_{t=0}^{\infty} \left( \frac{1}{1 + r} \right)^t [\pi (K_t) k_t - I_t - C(I_t) + q_t (k_{t-1} + I_t - k_t)]$$

The first-order condition for the firm’s investment in period $t$ is therefore

$$\frac{\partial L}{\partial I_t} = \left( \frac{1}{1 + r} \right)^t [-1 - C' (I_t) + q_t] = 0$$

which is equivalent to

$$q_t = 1 + C' (I_t)$$

(1)

The RHS is the cost of acquiring capital. The LHS is the value to a firm of an additional unit of capital at time $t$, measured in time-$t$ dollars, since it is the marginal value of relaxing the constraint $K_t \leq K_{t-1} + I_t$. The equation states that firms invest up to the point where they are equal.

Now the first-order condition for capital at time $t$ is

$$\frac{\partial L}{\partial k_t} = \left( \frac{1}{1 + r} \right)^t [\pi (K_t) - q_t + \frac{1}{1 + r} q_{t+1}] = 0$$

which, defining $\Delta q_t \equiv q_{t+1} - q_t$, is equivalent to

$$\Delta q_t = r q_t - (1 + r) \pi (K_t)$$

(2)

or

$$q_t = \pi (K_t) + \frac{1}{1 + r} q_{t+1}$$

$q_t$ is the value the firms attaches to a unit of capital at period $t$ in time-$t$ dollars, and $q_{t+1}$ is the value it will attach to a unit of capital at period $t + 1$ in time-$t + 1$ dollars. So $q_t$ has to be equal to the sum of
the contribution of the unit of capital to current profits plus its discounted value next period, otherwise the firm’s valuations would be inconsistent.

Equations (1) and (2) are key to understanding the evolution of the capital stock and the market price of capital in this model. We can see it on the phase diagrams. The continuous-time equivalents of these equations are

\[
q_t = 1 + C'(I_t) \\
rq_t = \pi(K_t) + \dot{q}_t
\]

- First look at equation (1). Since all firms are identical, they all take the same investment decision and \( \Delta K_t = N \Delta k_t = N * I_t \). Firms invest a positive amount if \( q > 1 \), and disinvest if \( q < 1 \). So in a \((K, q)\) diagram, \( K \) increases above the \( q = 1 \) line and decreases below it.

- Next, look at equation (2). It implies that the price of capital \( q \) rises if \( q_t > \frac{\pi(K_t)}{r} \) and falls if \( q_t < \frac{\pi(K_t)}{r} \). The profit function \( \pi(K_t) \) is decreasing in the aggregate capital stock, so the locus \( \dot{q} = 0 \) is downward sloping in a \((K, q)\) diagram.

- The initial \( K \) is given, and it evolves sluggishly as firms invest. The price \( q \) is free to adjust instantaneously. In this model, there is a unique saddle path, which means that for a given \( K \), there is a unique level of \( q \) such that \( K \) and \( q \) converge to a stable steady state.

- \( q \) can jump, but only in response to news. There cannot be an anticipated jump in asset prices, otherwise there would be an arbitrage opportunity.

**Example**: Assume an investment tax credit is introduced. Firms receive a rebate of a fraction \( s \) of the price of investment, not including adjustment costs.

1. How does this modify equations (1) and (2)?
2. If the ITC is permanent and was not anticipated, what is its effect on \( K \) and \( q \)?
3. If the ITC is temporary and was not anticipated, what is its effect on \( K \) and \( q \)?
4. If the ITC is permanent and had been announced before it is introduced, what is its effect on \( K \) and \( q \)?