MATHEMATICS 154, SPRING 2010
PROBABILITY THEORY
Outline #3 (Combinatorics, bridge, poker)

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References:

- PRP(Probability and Random Processes, by Grimmett and Stirzaker), Section 1.7.
- 1000Ex(One Thousand Exercises in Probability, by Grimmett and Stirzaker), page 135.
- Web pages on poker and bridge from www.durangobill.com

1. Principles of counting

   In the following items, use these principles of counting, taken for granted in PRP .

   - Principle 1: Multiply for sequential counting. If we are forming a set of n-tuples where there are k_1 choices for the first element x_1, k_2 for x_2, and so on, the number of n-tuples $x_1 x_2 ... x_n$ in the set is $k_1 k_2 ... k_n$
   - Principle 2: Divide to correct systematic overcounting. If M different n-tuples all correspond to the same element in our sample space, count the n-tuples and then divide by the number of times M that each was overcounted.
   - Principle 3: Divide and Conquer. To count the number of elements in a union of disjoint subsets, count each subset and sum the results.
   - Principle 4: Subtract off special cases. To count the number of elements in a difference $A \setminus B$ where $B \subset A$, count each set and take the difference.

   Classic example of principles 1 and 2: counting the number of k-element subsets of an n-element set.

   First count the k-tuples whose elements are all different:

   $$n(n-1)(n-2)...(n-k+1).$$

   Then divide by $k!$ to correct for overcounting. The result is the familiar "combinations" formula

   $$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Using binomial coefficients whenever possible is the safest way to avoid accidental overcounting.

For the following topics, the easiest way to get numerical answers is to use Maple or Mathematica. Any Harvard undergraduate can download a copy for free. For details, go to http://www.fas.harvard.edu/cgi-bin/software/download.pl and follow instructions.

2. Counting poker hands

   Count the number of ways to get each of the following types of 5-card poker hands, using a deck of 52 cards with 4 cards of each of 13 ranks, and use this to calculate the probability of being dealt such a hand.

   - 4 of a kind(four cards of one rank, the fifth of a different rank)
   - a full house(three cards of one rank, two of another)
   - 3 of a kind(three cards of one rank, two others of different ranks)

   <u>Solutions:</u>

   - How many distinct 5-card poker hands can be dealt from a 52-card deck?
     Answer: Select 5 cards sequentially: the number of ways to do this is $52 \times 51 \times 50 \times 49 \times 48$
     But this generates each distinct hand $5! = 120$ times.
     So the number of distinct hands is $\binom{52}{5} = \frac{52!}{47!5!} = 2,598,960$
   - How many distinct ways are there to get 4 of a kind, and what is the probability?
     13 choices for the rank of the 4, 48 choices for the other card.
     $$\mathbb{P} = \frac{13 \cdot 48}{2,548,960} = 0.000240096$$
   - A full house (3 of one rank, 2 of another)
     13 choices for the first rank (with 3 cards), 4 for the suit that is missing.
     12 choices for the second rank (with 2 cards), $\binom{4}{2} = 6$ for the pair of suits that have the second rank.
     $$\mathbb{P} = \frac{13 \cdot 4 \cdot 12 \cdot 6}{2,548,960} = 0.0014405762$$
   - Three of a kind (3 of one rank, the other two are of different ranks)
     13 choices for the first rank (with 3 cards), 4 for the suit that is missing.
     $\binom{12}{2} = 66$ choices for the pair of ranks of the other two cards; 16 choices for the suit of each. So the number is $13 \cdot 4 \cdot 66 \cdot 16 = 54,912$ and
     $$\mathbb{P} = \frac{54,912}{2,548,960} = 0.0211284514$$

3. Bridge problems

   A bridge hand consists of 13 cards from a 52-card deck.

   - Count the number of bridge hands with 6 spades, 4 hearts, 2 diamonds, and 1 club.
   - Count the number of bridge hands with 6-4-2-1 suit distribution (6 cards in the longest suit, 4 in the second-longest, 2 in the third-longest)
   - Count the number of bridge hands with 4-4-3-2 or 4-3-3-3 suit distribution, and show that the former has a higher probability.

<u>Solution:</u>

- How many ways are there to select 6 cards from the 13 spades?

$$\binom{13}{6} = \frac{13!}{7!\,6!}$$

- How many distinct hands have 6 spades, 4 hearts, 2 diamonds, 1 club?
  Apply the same analysis to each suit in turn:

$$\binom{13}{6}\binom{13}{4}\binom{13}{2}\binom{13}{1} = \frac{13!}{7!\,6!}\frac{13!}{9!\,4!}\frac{13!}{11!\,2!}\frac{13!}{12!\,1!}$$

- How many distinct hands have a 6-4-2-1 distribution?
  Multiply the preceding number by the number of ways to choose the 6-card suit, the 4-card suit, etc, which is 4!=24.
- How many distinct hands have a 4-4-3-2 distribution?
  There are 4 ways to select the 2-card suit followed by 3 ways to select the 3-card suit, so we will multiply by 12.

$$N_{4432} = 12\binom{13}{4}\binom{13}{4}\binom{13}{3}\binom{13}{2} = 12\frac{13!}{9!\,4!}\frac{13!}{9!\,4!}\frac{13!}{10!\,3!}\frac{13!}{11!\,2!}$$

- How many distinct hands have a 4-3-3-3 distribution?
  Now there are 4 ways to select the 3-card suit, so we will multiply only by 4.

$$N_{4333} = 4\binom{13}{4}\binom{13}{3}\binom{13}{3}\binom{13}{3} = 4\frac{13!}{9!\,4!}\frac{13!}{10!\,3!}\frac{13!}{10!\,3!}\frac{13!}{10!\,3!}$$

  The ratio is

$$\frac{\mathbb{P}_{4432}}{\mathbb{P}_{4333}} = \frac{N_{4432}}{N_{4333}}\frac{12\;10!\,3!\,10!\,3!}{4\;9!\,4!\,11!\,2!} = \frac{45}{22}$$

4. Dividing up a set with two type of objects

   Suppose we have a set S with $2n$ elements, m of one type and $2n - m$ of the other. As a concrete example, suppose that Lisa has $2n = 6$ Dunkin' Munchkins, of which $m = 2$ are chocolate and the remaining $2n - m = 4$ are plain. She chooses $n = 3$ of them at random and puts them in a bag for her son Thomas. The remaining three go into a bag for her daughter Catherine.

   In general we divide up S at random into set S_1 and set S_2, each with n elements. The number of different ways of doing this is equal to the number of ways of selecting the n elements of S_1, namely

$$N = \binom{2n}{n} = \frac{(2n)!}{n!n!}$$

   We are interested in the probability that k of the "special objects" (chocolate Munchkins) end up in set S_1. Assume that $k \leq m$, $k \leq n$, and $k \geq m - n$. The number of ways of selecting k special objects and $n - k$ non-special objects for set 1 is

$$M = \binom{m}{k}\binom{2n-m}{n-k} = \frac{m!}{k!(m-k)!}\frac{(2n-m)!}{(n-k)!(n+k-m)!}$$

   If we assume that each way of selecting the elements of S_1 is equally likely, the ratio $\frac{M}{N}$ gives the probability that set S_1 contains precisely m special objects.

   As a practical matter, it is much easier to repeat this reasoning in special cases than to memorize these formulas!

5. Waring's Theorem

This useful result is proved cleverly in chapter 5 of PRP. Here is a straightforward proof inspired by the solution to problem 13 on page 22 of PRP (solutions on page 143 of 1000Ex). It avoids the use of the result from problem 12.

We have n events A_i, $i = \{1, 2, ...n\}$.

Examples:

- Toss 4 coins; A_i is "the ith coin comes up heads."
- Place 5 cups of different colors randomly on 5 colored saucers; A_i is "the ith cup (blue) lands on the blue saucer."
- Buy 6 boxes of cereal, each containing a bust of a recent president; A_i is "at least i of your 6 busts is a Barack Obama."

Let S be a subset of $\{1, 2, ...n\}$ like

- the first and fourth coin tosses
- the red, blue, and yellow cups
- Clinton and Bush

Define events: A_S = "all A_i for $i \in S$ occur, and maybe others too."

B_S = "no event A_i for i outside S occurs."

So $A_S \cap B_S$ = "all A_i for $i \in S$ occur, and no others." This is one specific way that precisely k events can occur. For different choices of S, such events are disjoint. Sum over S to get the probability of their union.

N_k = "precisely k events occur."

In many cases, $\mathbb{P}(A_S)$ is easy to calculate, but $\mathbb{P}(N_k)$ is a pain to determine. **Waring's theorem** relates the two:

$$\mathbb{P}(N_k) = \sum_{|S|=k} \mathbb{P}(A_S) - \binom{k+1}{k} \sum_{|T|=(k+1)} \mathbb{P}(A_T) + \binom{k+2}{k} \sum_{|T|=(k+2)} \mathbb{P}(A_T) - \cdots$$

Proof:

Start, as usual, with a disjoint union:

$$\mathbb{P}(A_S) = \mathbb{P}(A_S \cap B_S) + \mathbb{P}(A_S \cap B_S^c)$$

$$\mathbb{P}(A_S \cap B_S) = \mathbb{P}(A_S) - \mathbb{P}(A_S \cap \bigcup_{j \notin S} A_j)$$

$$\mathbb{P}(A_S \cap B_S) = \mathbb{P}(A_S) - \mathbb{P}(\bigcup_{j \notin S} (A_S \cap A_j)).$$

Now apply generalized inclusion-exclusion to the union in the right term:

$$\mathbb{P}(A_S \cap B_S) = \mathbb{P}(A_S) - \sum_{j \notin S} \mathbb{P}(A_S \cap A_j) + \sum_{j<k \notin S} \mathbb{P}(A_S \cap A_j \cap A_k) - \cdots$$

Finally, sum over all subsets of $\{1, 2, ...n\}$, for sets of size $k, k+1, k+2, \cdots, n$ to get the theorem:

$$\mathbb{P}(N_k) = \sum_{|S|=k} \mathbb{P}(A_S \cap B_S) =$$

To avoid confusion, I used the generic symbol T for sets with more than k elements. Each set T with $k+m$ elements will arise in the sum $\binom{k+m}{k}$ times, once from each of its k-element subsets that can serve as S.

For the problem of placing R, B, Y, and G cups on 4 matching saucers, the probability of getting exactly one cup on the saucer of the same color is, according to Waring's theorem,

$$\mathbb{P}(N_1) = 4\mathbb{P}(A_1) - 6\binom{2}{1}\mathbb{P}(A_1 \cap A_2) + 4\binom{3}{1}\mathbb{P}(A_1 \cap A_2 \cap A_3) - \binom{4}{1}\mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4)$$

$$\mathbb{P}(N_1) = 4\frac{1}{4} - 12\frac{1}{12} + 12\frac{1}{24} - 4\frac{1}{24} = 1 - 1 + \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

6. Waring's theorem applied to coin tossing

   - What is the probability of getting exactly one head when you toss three fair coins?

     Fundamental events:

     A_i is the event that coin i comes up heads.

     Clearly $\mathbb{P}(A_i) = \frac{1}{2}$. Since the tosses are independent events, $\mathbb{P}(A_i \cap A_j) = \frac{1}{4}$ for $i \neq j$, and $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \frac{1}{8}$.

     Since the probability of all heads is the same for any set of coins of a given size, we need only consider probabilities like $\mathbb{P}(A_1)$ and $\mathbb{P}(A_1 \cap A_2)$. The summation required by Waring's theorem can be accomplished by multiplying by the number of relevant events.

     N_1 is the event that exactly one coin comes up heads. Waring says

$$\mathbb{P}(N_1) = 3\mathbb{P}(A_1) - 3\binom{2}{1}\mathbb{P}(A_1 \cap A_2) + \binom{3}{1}\mathbb{P}(A_1 \cap A_2 \cap A_3).$$

$$\mathbb{P}(N_1) = 3\frac{1}{2} - 6\frac{1}{4} + 3\frac{1}{8} = \frac{3}{8}.$$

   - What is the probability of getting exactly two heads when you toss three fair coins?

     N_2 is the event that exactly two coins comes up heads. Waring says

$$\mathbb{P}(N_2) = 3\mathbb{P}(A_1 \cap A_2) - \binom{3}{2}\mathbb{P}(A_1 \cap A_2 \cap A_3).$$

$$\mathbb{P}(N_1) = 3\frac{1}{4} - 3\frac{1}{8} = \frac{3}{8}.$$

7. Waring's theorem applied to cards

   What is the probability of getting exactly one spade when you select two cards from a shuffled deck?

   A_i is the event that card i is a spade. In this case $\mathbb{P}(A_i) = \frac{1}{4}$ for each card. However,

   $\mathbb{P}(A_1 \cap A_2) = \frac{13 \cdot 12}{52 \cdot 51} = \frac{1}{17}$.

   N_1 is the event that exactly one card is a spade. Waring says

$$\mathbb{P}(N_1) = 2\mathbb{P}(A_1) - \binom{2}{1}\mathbb{P}(A_1 \cap A_2).$$

$$\mathbb{P}(N_1) = 2\frac{1}{4} - 2\frac{1}{17} = \frac{13}{34}.$$

8. Waring's theorem applied to dice

What is the probability of getting exactly one 6 when you roll three fair dice?

A_i is the event that die i comes up 6.

Clearly $\mathbb{P}(A_i) = \frac{1}{6}$. Since the tosses are independent events, $\mathbb{P}(A_i \cap A_j) = \frac{1}{36}$ for $i \neq j$, and $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \frac{1}{216}$.

Since $\mathbb{P}(A_1) = \mathbb{P}(A_2) = \mathbb{P}(A_3)$ and $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_2 \cap A_3)$, we need only consider probabilities like $\mathbb{P}(A_1)$ and $\mathbb{P}(A_1 \cap A_2)$. The summation required by Waring's theorem can be accomplished by multiplying by the number of relevant events.

N_1 is the event that exactly one die comes up 6. Waring says

$$\mathbb{P}(N_1) = 3\mathbb{P}(A_1) - 3\binom{2}{1}\mathbb{P}(A_1 \cap A_2) + \binom{3}{1}\mathbb{P}(A_1 \cap A_2 \cap A_3).$$

$$\mathbb{P}(N_1) = 3\frac{1}{6} - 6\frac{1}{36} + 3\frac{1}{216} = \frac{75}{216}.$$

N_2 is the event that exactly two dice come up 6. Waring says

$$\mathbb{P}(N_2) = 3\mathbb{P}(A_1 \cap A_2) - \binom{3}{2}\mathbb{P}(A_1 \cap A_2 \cap A_3).$$

$$\mathbb{P}(N_1) = 3\frac{1}{36} - 3\frac{1}{216} = \frac{15}{216}.$$

N_0 is the event that no die comes up 6. Waring says

$$\mathbb{P}(N_0) = 1 - 3\binom{1}{0}\mathbb{P}(A_1) - 3\binom{2}{0}\mathbb{P}(A_1 \cap A_2) + \binom{3}{0}\mathbb{P}(A_1 \cap A_2 \cap A_3).$$

$$\mathbb{P}(N_0) = 1 - 3\frac{1}{6} + 3\frac{1}{36} - \frac{1}{216} = \frac{125}{216}.$$