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1 Introduction

Let’s summarize what we have done so far.

We have described the constraints that a consumer faces, i.e., discussed the budget constraint.

In many cases, this constraint is simple, whether when seen as an equation or as a graph. But in others, it gets messier.

We have also discussed a way of modeling what the consumer wants. We referred to this as preferences, and while we have only described them graphically, they can also be described algebraically, sometimes in a simple way, as we will see.

So far we have said nothing about what the consumer is going to choose; that is, we have not modeled the problem the consumer faces, although it has obviously been implied, and you may have seen it in other classes. We won’t do that for another lecture or so.
Instead, we are next going to discuss another way of describing the consumer’s desires / preferences.

This approach is known as the utility function approach; we are going to think about each possible consumption bundle as giving the consumer some amount of “utility,” whatever that is. We’ll define it more explicitly shortly.

The early versions of consumer theory focused on utility, without any mention of preferences, and assumed we could assign a unique utility number to every choice or bundle. These utility numbers had some sort of independent meaning. In particular, we could add and subtract them across people.

Modern economic theory does not proceed in this way. It focuses instead on preferences, and simply assumes that given various bundles, consumers can rank them and make choices.

Utility turns out to be, usually, a convenient way to summarize these preferences, but it is the preferences that are fundamental.

The reason we are going to move to utility functions from preferences is that they are easier to work with. But it is useful to understand that they are less fundamental than preferences per se.

In particular, the utility numbers have no cardinal meaning; they simply indicate that, if one bundle has higher utility number than another, then the one is preferred to the other. The actual value of the utility numbers is irrelevant, sort of like rankings at a golf tournament.

Note also that economists use both the preference approach and the utility function approach. We want to understand the relation between them. For many purposes, the two approaches are equivalent. Also, in many settings, it is not obvious why ones needs to have bothered with the preference approach. In a few crucial cases, however, it matters a lot, and that is why we need to do both.

We’ll see this more explicitly later in the course.
2 Utility: A Definition

A utility function is a way of assigning numbers to every possible consumption bundle in such a way that more-preferred bundles get assigned larger numbers than less-preferred bundles.

That is, $X$ is preferred to $Y$ if and only if the utility of $X$ is larger than the utility of $Y$. In symbols,

$$(x_1, x_2) > (y_1, y_2) \text{ iff } u(x_1, x_2) > u(y_1, y_2).$$

This relies only on the ordering of the bundles, not the absolute numbers.

The function $u$ is referred to as an ordinal utility function.

To illustrate, consider a consumer who prefers the bundle $X$ to the bundle $Y$ and the bundle $Y$ to the bundle $Z$.

<table>
<thead>
<tr>
<th></th>
<th>$U_1$</th>
<th>$U_2$</th>
<th>$U_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>100</td>
<td>8</td>
<td>-7</td>
</tr>
<tr>
<td>$Y$</td>
<td>10</td>
<td>.6</td>
<td>-82</td>
</tr>
<tr>
<td>$Z$</td>
<td>1</td>
<td>0</td>
<td>-1111</td>
</tr>
</tbody>
</table>

The numbers in each column are the utilities assigned to each bundle by three possible utility functions.

The point of this comparison is that each of the three set of numbers is a valid utility function.

Since only ranking matters, a unique way to assign utilities does not exist.

If one utility function exists that corresponds to a particular set of preferences, then an infinite number exists.

Consider $u(x_1, x_2)$. Then $2u(x_1, x_2)$ gives same ranking.

Indeed, any monotonic, increasing transformation gives the same ranking.
3 Monotonic Transformations

A monotonically increasing transformation is a function \( f \) with the property that it preserves the order of numbers. That is, if \( f \) is a monotonically increasing function, and

\[ u_1 > u_2 \]

then

\[ f(u_1) > f(u_2) \]

Examples of monotonic functions include the following:

- Multiplication by a positive number
- Adding a positive number
- Taking the ln
- Exponentiation
- Raising to an odd power
- Raising to an even power for functions defined over non-negative values
- Step functions, assuming they are increasing

Examples of functions that are NOT monotonically increasing include:

- Sin or cosine
- Squaring if the function is defined over negative values
- Multiplying by a negative

Mathematically, \( f \) always has a positive derivative, assuming it is differentiable. More generally, \( f \) always has a positive difference.
Graph: Monotone and Non-Monotone Functions

Monotone: $y = x^2$ for $0 \leq x$

Non-monotone: $y = \sin 2x + \cos 3x + 5$

So, a key result we want is the following:
If $u$ is a utility function and $f$ is monotone, then $f(u)$ is a utility function.

Why?

1. Saying $u$ is a utility function corresponding to some preferences means
   
   \[ u(X) > u(Y) \text{ iff } X \succ Y \]

2. If $f$ is monotone, then
   
   \[ u(X) > u(Y) \text{ iff } f(u(X)) > f(u(Y)) \]

3. Therefore,
   
   \[ f(u(X)) > f(u(Y)) \text{ iff } X \succ Y \]

so the new function $f(u())$ represents the preferences in the same way as the original $u$.

Geometrically, a utility function is a way of labeling the indifference curves. Every bundle on same curve has the same $u$. A higher indifference curve has a higher number (assuming monotonicity of preferences).

A monotone transform is just a relabeling that preserves the order of the utility numbers.

4 Cardinal Utility

We can, in principle, assign cardinal utilities.

The difference in a consumer’s preference for particular bundles could be based on how much she is willing to pay for the bundle, how far she needs to walk to get the bundle, how long she has to wait in line, etc.

These are defensible interpretations, but none is compelling.

Also, in the context of an individual’s behavior, they do not add anything. We can predict an individual’s behavior based on the ordinal numbers by themselves.
Knowing how much more a consumer values one bundle relative to another does not tell us anything useful.

We tend to think we want cardinal utilities to do interpersonal comparisons. We will come back to that later in the course, it's an important issue.

5 Constructing a Utility Function

Now, let’s try to make the basic concept clearer.

Let’s turn to the question of how we might assign ordinal utilities, given that we have some well-defined preferences.

The key question is whether we can always find a utility function that correctly represents those preferences?

In the most general case, no. Say someone prefers A to B to C to A. Then a utility function representing those preferences would have to satisfy $u(A) > u(B) > u(C) > u(A)$, which is impossible.

So, we do have to adopt some restrictions, such as transitivity, to assure a utility function exists for a given set of preferences.

(This shows why the utility function approach is less general than just using preferences, although it’s not that much less general.)

Start with the indifference curve map. We in essence want a way to label the indifference curves so that higher curves get higher numbers.

Here’s a way: draw a diagonal line and label each point on the line with its distance from the origin.

If preferences are monotonic, the line through the origin must intersect every indifference curve exactly once.

Why? Because by assuming monotonicity, we have ruled out bads and satiation.

So, every bundle is getting exactly one label. That’s all it takes.

There might be other ways to assign utility (assuming monotonicity). But this example shows there is always one way, assuming “well-behaved” preferences.
6 Examples of Utility Functions

Now let’s look at examples.

6.1 Indifference Curves from Utility Functions

Suppose you are given the utility function $u(x_1, x_2) = x_1 x_2$. Take as given for the moment that this is a well-defined utility function. What do the indifference curves look like?

To do this, note that the indifference curve associated with a particular level of utility is the set of all $(x_1, x_2)$ such that

$$u(x_1, x_2) = \bar{u},$$

where $\bar{u}$ is a constant. This is known as a level set. Thus, for each constant, we get a different indifference curve.

In this case,

$$u(x_1, x_2) = x_1 x_2 = \bar{u}$$

implies

$$x_2 = \frac{\bar{u}}{x_1}$$
Graph: Indifference Curves for the Hyperbolic Level Set

\[ x_2 = \frac{\bar{u}}{x_1} \]
\[ x_2 = \frac{3}{x_1} \]
\[ x_2 = \frac{5}{x_1} \]
\[ x_2 = \frac{7}{x_1} \]

A different example:

\[ v(x_1, x_2) = x_1^2 x_2^2 \]

This is just the square of the previous one.

The original one cannot be negative. Thus, over the range of positive \( x_1 \) and \( x_2 \), the square is a monotonic transformation.

Given that it is a monotonic transformation of the original utility function, these must be the same preferences.

Therefore, we should have the same indifference curves, but with different labels.

For example, 1, 2, 3 corresponds to 1, 4, 9 now.
\[ x_2^2 = \frac{u}{x_1^2} \]

That is, if with the original labeling, we had, say
\[ \bar{u}_0 = x_1x_2 \]
and we now have
\[ \bar{u}_1 = x_1^2x_2^2 \]
for the same \( x_1 \) and \( x_2 \), then squaring the first one implies that
\[ \bar{u}_0^2 = \bar{u}_1 \]
So, if \( \bar{u}_0 \) had label of 3, then \( \bar{u}_1 \) has a label of 9. Thus, this utility function orders bundles the same way.

This is an example in one direction: given a utility function, determine the indifference curves.

### 6.2 Utility Functions from Indifference Curves

We also might want to go other way: given the preferences / indifference curves, find a corresponding utility function. We can often do it mathematically. Here, we will just figure it out intuitively in some interesting cases.

#### 6.2.1 Perfect Substitutes

In the case of preferences that regard two different goods as perfect substitutes, the only thing a consumer cares about is the (possibly weighted) number of units of the two goods.

So, consider utility functions of the form
\[ u(x_1, x_2) = ax_1 + bx_2 \]
This says the consumer would give up 1 unit of $x_1$ for $a/b$ units of $x_2$, no matter what the initial levels are. Or, in other words, the consumer would give up $b$ units of $x_1$ for $a$ units of $x_2$. The rate of trade-off is a constant, which is what we mean by perfect substitutes.

This formula also gives a higher level of utility to bundles with more units.

If $a = b = 1$, this just says that the consumer cares only about the number of goods, not the composition; thus, the goods are perfect substitutes.

The slope of the indifference curves is $-a/b$.

We could also use, of course, the square of $ax_1 + bx_2$.

### 6.2.2 Perfect Complements

A utility function that describes perfect complements is

$$u(x_1, x_2) = \min\{ax_1, bx_2\}$$

Why? This is because if you always want to consume these goods in exact proportions, then anything beyond this proportion for one good, but not the other, adds nothing to utility.

Take the case where $a = b = 1$ (e.g. right shoes and left shoes).

The utility you get from $(2, 1)$ or $(1, 2)$ or $(1, 1)$ is the same. Likewise for $(50, 1), (1, 50)$, etc. All that matters is the good of which you have the least, i.e., the minimum.

If $a$ or $b$ is not one, this becomes mildly tricky.

Say we are considering gin and vermouth, which you only consume in martinis in the fixed proportion of 5 gin to 1 vermouth.

Then the utility function is

$$u(g, v) = \min\{(1/5)g, v\}$$
Unless we have 5 times as many units of gin as vermouth, we do not get any extra utility.

Any monotonic transformation works as well. For instance, multiplying by 5 to get rid of the fraction:

\[ u(g, v) = \min\{g, 5v\} \]

6.2.3 Quasilinear Preferences

Imagine that the consumer’s indifference curves are just vertical shifts, as in graph:
We can show that this must mean
\[ x_2 = k - v(x_1) \]
where \( k \) is a different constant for each indifference curve. A higher \( k \) means a higher indifference curve.

A natural way to label the indifference curves is with \( k \).

Solving for \( k \) and setting it equal to utility gives
\[ u(x_1, x_2) = k = v(x_1) + x_2 \]
So, the utility function is linear in good 2, but not necessarily in good 1. Hence the name quasilinear.

Examples:
\[ x_1^5 + x_2 \]
\[ \ln x_1 + x_2 \]

These functions are easy to work with, even if they are not always realistic.

### 6.2.4 Cobb-Douglas Preferences

Another common utility function is Cobb–Douglas:

\[ u(x_1, x_2) = x_1^c x_2^d \]

where \( c \) and \( d \) are positive numbers that describe preferences.

The general graphical shape looks like:
Cobb-Douglas utility functions correspond to convex, monotonic preferences.

Once again, monotonic transformations also work:

\[ c \ln x_1 + d \ln x_2 \]

Or, raising \( x_1^c x_2^d \) to the power

\[ 1/(c + d) \]

implies

\[ x_1^a x_2^{1-a} \]

where

\[ a = c/(c + d) \]

This is useful for future examples.
7 Preview: Marginal Utility and the Marginal Rate of Substitution

We have defined and described utility functions, looked at examples, and related them to indifference curves.

We have not mentioned marginal utility or the marginal rate of substitution; the order in which I am doing things is a bit different than the text.

We will get to that when we examine the actual choice problem later.