Quantum Entanglement and Tensor Network RG

For many decades, the Landau theory of symmetry breaking has dominated thinking about phase transitions. Recent interest in topological phases and quantum orders, which lie beyond the realm of symmetry breaking theory. Need a more robust framework. RG provides. I describe how numerical RG can be used to understand phases beyond sym. breaking. Implementation for quantum ordered phases requires introduction of a new approach – entanglement filtered tensor network RG – that I will introduce in this talk. There is hope that using these methods, all strongly correlated systems could be understood, at least numerically. Tall order, but here is the state of play. Outline 1) Overview of topological and quantum orders 2) The RG approach to Phases, and problems. 3) Tensor Network RG 4) Entanglement and entanglement filtering 5) Tensor Network RG applied to spin 1 chain and Haldane Phase 6) Looking ahead.

1) What is the problem we want to solve?
Traditionally think about phase transitions in terms of Landau symmetry breaking theory – ordered and disordered state characterized by local symmetry order parameter. However, there are other states of matter. Topological phases (eg FQHE, spin liquids, superconductors, even) are states that exhibit long range order, but do not break any local symmetries. They are gapped, and contain ground state degeneracies of topological origin i.e. robust against perturbations that break all the symmetries of the Hamiltonian. E.g. FGHE degeneracy depends only on filling fraction of FQHE state and topology of manifold on which theory is formulated (sphere, torus etc). Experimental signature is chiral edge states with linear dispersion. Can be classified using slave boson formalism
and projective symmetry group, which I will not discuss here. But all this is rather ad hoc, and PSG is formulation dependent, which is unsatisfactory. It would be nice to have a more robust approach to dealing with topological phases. Also, there is the problem of quantum orders. Hypothetical gapless phases with long range order but no broken symmetry. Somehow connected to the pattern of long range quantum entanglement in the state. Very little understood about them besides the name and this fact. Therefore, we really need a more robust way to think about phases, that can deal with topological phases and quantum orders. RG provides such an approach.

2) The RG approach to Phase Structure

Begin with observation that quantum partition function contains all information about a system, therefore, if we know partition function, we know all the phase structure. We know how to calculate quantum partition function as functional field integral. Therefore, calculate partition function in parameter space and job done. Singularities are phase transitions, separate different bulk phases. Two bulk phases sharply distinct iff cannot smoothly deform from one to other without crossing phase transition (counterexample: not liquid/gas). Requires us to specify what deformations are allowed (symmetries of / constraints on the Hamiltonian). Only way to calculate integral is perturbatively, via RG. Each bulk phase is controlled by a stable fixed point under RG. Therefore, phase structure can be completely characterized by a) symmetry group of microscopic Hamiltonian (needed to specify which deformations are allowed, and for symmetry breaking phases to make sense), and b) fixed point Lagrangian (properties of phase). Robust, and capable of dealing with anything. But there is a problem…
Consider simple spin models. Can be recast as bosonic theories via Holstein Primakov. However, believe exhibit topological phases (chern Simons field theories) and quantum orders (emergent fermions and gauge particles). And if we do perturbative coupling constant RG, we can only flow to another bosonic theory of same sort. So approach is breaking down. Why? Momentum shell RG makes assumption that free theory looks simple and diagonal in momentum space (e.g. $k < k_F$ filled, $k > k_F$ empty). This is appropriate and useful for weakly interacting systems, and allows $k$ space RG, but is not appropriate for strongly interacting systems. Plus, large $k$ modes are not high energy modes. We have no idea what a strongly interacting ground state looks like in momentum space, so we don’t even know where to start. Better to formulate in real space, and to introduce…

3) tensor network RG.

Want to formulate RG in real space (it will turn out entanglement is local and therefore easier dealt with in real space). UV cutoff? Lattice. So integral over space -> sum over lattice sites (note no continuum approximation need be invoked). Since real space lattice RG can only be done numerically, we also discretise the states (OK even in principle). To each lattice site, assign a physical index specifying state lattice site is in and contribution to partition function. If sites decoupled, easy. Sum over one site, raise to power $N$. real lattice not decoupled. $Nn/ nnn/ lr$ coupling. Therefore introduce virtual indices that control contractions. For $nn$ coupling, $z$ virtual indices. Can represent exact partition function as trace over tensor network. Choice of tensor = choice of Lagrangian. Example of Ising case with spins on plaquettes ($T_{1111} = T_{2222} = e^{+(+4\beta)}$, $T_{1212} = e^{(-…}$
4beta)=T2121. Others 1.) Hilbert space D^N dimensional. Too large to deal with, therefore do RG. Tensors flow to fixed point. Note tensors could in principle flow to anything (not bosonic to bosonic), therefore have introduced extra degree of freedom into theory. Guess, - what form of fixed point tensor? In high T phase, beta -> zero therefore all tensors 1, which can be rotated into isotropic tensor T1111 = 1, all other 0. (Fixed point tensors unique only upto change of basis, with allowed changes of bases limited by symmetry group). Isotropic tensor. Good. Symmetry broken state flows to 1 * 1 (also good (want all up or all down, nothing else)). So in this case can guess fixed point tensors. In general, disorders phases have fixed point kronecker delta (no structure), whereas broken symmetry phases are block diagonal and topological phases are not block diagonal (explain: non trivial because ordered, not block diagonal because they contain admixtures of all symmetries). How to do RG?

This is how. Explain, with diagrams. In principle – contract, truncate, contract, truncate… four at a time in 1+1D. (This is a numerical procedure). In practice, easier to do two at a time. Rewire tensor network to two at a time. (draw picture). Contract two tensors. D ->: D^2. Now truncate (keep a D dimensional subspace of the full Hilbert space, throw away to rest). Want to choose subspace to keep so as to minimize truncation error. How do this? Use SVD method. Trace is basis invariant, so diagonalise T’, keep D largest eigenvalues and throw away the rest. (Can see topological phases, with degenerate eigenvalues, will resist simplification therefore will not be reduceable). Anyway, SVD based truncation allows us to coarse grain. Explain. Questions? Iterate. This is how to do tensor network RG. It is obvious this will not work. Why? Is real space equivalent of k
space RG, which was bust because of entanglement anyway. So how to deal with this? Well, a bit of an aside, but must first understand the problem, and the problem is…

4) Entanglement.

What is entanglement? Simple example: Bell states \((00 + 11)\) with correlated (non independent) measurement outcomes. If we measure the first qubit to be in state 0, the other must be also. As a concept, extremely pervasive. Quantum information, black holes (Beckenstein Hawking) and now many body Physics. But what is entanglement in a many body system? Hard to picture, but we can quantify it. Introduce concept of entanglement entropy \(= \text{Tr} \, p_A \ln p_A\) where \(p_A\) is density matrix is subsystem. If zero, the in pure state, no entanglement. The bigger the entanglement entropy, the more entangled. Typically, entanglement entropy follows an area law (proportional to area and to correlation length). Therefore, less of a problem in 1D, away from criticality. Can even be measured in 1d using noise patterns. So we may not know what it is, but we can quantify it and measure it. And we can understand the problem – we were picking out the wrong Hilbert space to keep, because we weren’t taking into account correlations with surrounding sites. How to deal with it? One simple way – truncate subject to minimization of entanglement entropy (implemented using Lagrange multiplier). Conceptually very elegant, but computationally very expensive. Computationally, preferable to use the superblock methods first introduced by White. Key to these is to delay truncation. Illustrate in 1D.
Let us say we want to coarse grain two sites 0 and 1. Compact together a preset number m sites (on either side) into a Hilbert space of size D^m (exact). Then do SVD. Now take the D largest eigenvalue states, and project them onto 0,1. Now replade the D^2 dimensional Hilbert space for 0,1 by the D dimensional Hillbert space obtained by projection of superblock eigenstates onto 0,1. Net result is the same – coarse graining of 0,1 site to D dimensional. But choice of which subspace to keep has now incorporated the effects of sites upto m/2 away. This step is known as entanglement filtering – i.e. taking into account the contribution of neighbouring sites before deciding which states to keep. Works best if m is an appreciable fraction of correlation length (i.e. not too close to criticality, except in 1+1D where entanglement is relatively unimportant). After this coarse graining, superblock is m-1 sites. Therefore expand one site and repeat. Iterate. Add one site at a time normally more efficient. This sort of filtering process usually allows us to filter out short range entanglement (which screws up RG), while retaining long range entanglement (which is the hallmark of topological and quantum orders). Armed with TEFRG, we can tackle real problems. Many examples in paper, but I will talk about just one

5) Spin 1 chains and Haldane gap.

Consider a 1D spin 1 chain. Hamiltonian Si S(i+1) + U(Sz)^2 + BSx. Has time reversal, parity, Sy Z2 symmetry. Do usual path integral construction e^(-delta tau H_i), and take matrix elements between spacetime slices. It turns out that the matrix elements can be expressed as rank 4 tensors placed on the links therefore have tensor network trace over network of rank 4 tensors. Do TEFRG. Use anisotropic coarse graining for technical
reasons (to reduce truncation errors). It emerges there are 4 phases. The trivial disordered phase, two symmetry breaking Z2 phases Z2x and Z2y (these have fixed point tensors with the same structure, but cannot be mapped onto one another because the transformation that turns one fp tensor into the other violates the symmetries of the Hamiltonian). Finally, there is a tensor described by a rank 4 tensor. Large ambiguity in form of fp tensor. This represents Haldane phase. Haldane phase has fourfold degeneracy (and in Haldane phase tensors flow to degenerate eigenvalues). Therefore SVD simplification based on eigenvalue based truncation fails. Is this phase distinct to the trivial (also gapped) phase? Yes, if the Hamiltonian obeys only translation symmetry (in this case the Haldane phase is trivial). However…

Numerical calculation indicates that Haldane phase is robust against any perturbations provided that they respect T and P symmetry (new result from TEFRG). Therefore, Haldane phase is distinct to trivial phase is T,P symmetry observed. Commonly called symmetry protected topological phase, but we can’t really have topological phases in 1D (topology of a line is trivial), so should more properly be called a symmetry protected distinguishability of two trivial phases in 1D. The stability of the Haldane phase in presence of T,P violation is principal result of paper. Earlier attempts to classify Haldane phase by means of boundary spin degeneracy suggested magnetic field destroys Haldane phase. Not so. New result. Can obtain low energy excitations from fixed pt tensor. In 1+1D, can also extract CFT data on phase transition using numerical RG (like central charges). This because entanglement is less bad in 1+1D. In higher dimensions, region near criticality much more difficult to deal with. Can even try to infer ground state
wavefunction from fp tensor. Rank 4, therefore 4 degrees of freedom per site. Can think of this in terms of two spin halves, which suggests spin half dimer state. OK last is a stretch, but still, points for trying.

6) Looking ahead.

We’ve covered a lot or ground, so let us recapitulate. We saw that we wanted to go beyond Landau theory. Objective was to understand topological phases (gapped phases with topological degeneracy, long range order, and no symmetry breaking) and quantum orders (gapless ordered non symmetry breaking states), which could not be described in terms of symmetry breaking. We hoped that the RG structure would allow us to robustly classify and understand all phases in terms of bare Hamiltonian and fixed point lagrangian. Problem was how to implement RG for topological phases and quantum orders. Introduced the tensor network RG, in which partition function is written as a tensor network trace, and tensors are coarse grained by SVD projection. We saw problem – entanglement, and discussed what entanglement was. Then discussed a method for dealing with entanglement – superblock method, and applied resulting TEFRG to spin 1 chain, and learned something new about Haldane phase. Well, OK, interesting, and a lot of work – but what is it good for? Well, in principle, could solve all strongly correlated systems numerically. Much more powerful than traditional RG because it allows the lagrangian (tensor) to change form. OK, numerically, but still, that’s pretty good.

Remaining problems are largely numerical. How to deal with larger dimensions, where entanglement is more of a problem? How to deal with criticality? The filtering algorithm introduced was very crude. Need to do better. Exciting and promising approach, but a work still in progress. Special thanks to Brian Swingle, Xiao Gang Wen and Zcheng Cheng Gu for answering questions. Hope you found that illuminating. Thank you.