Lecture 13: Path Integrals

1 Introduction

We have learned quantum field theory using the canonical quantization approach, which is based on creation and annihilation operators. There is a completely different way to do QFT called the Path Integral formulation. It says

$$\langle 0|T\{\phi(x_1),\ldots,\phi(x_n)\}|0\rangle = \int D\phi \phi(x_1)\ldots\phi(x_n)e^{iS[\phi]} \quad (1)$$

The left-hand side is exactly the kind of time-ordered product we use to calculate $S$-matrix elements. The $D\phi$ on the right-hand side means integrate over all possible classical field configurations $\phi(x,t)$ with a phase given by the classical action evaluated in that field configuration.

The intuition for the path integral comes from a simple thought experiment you can do in quantum mechanics. Recall the double slit experiment: the amplitude for field to propagate from a source through a screen with two slits to a detector is the sum of the amplitudes to propagate through each slit separately. We add up the two amplitudes separate and then square to get the probability. Now try three slits, you get the sum of 3 paths. Now, keep slitting until the screen is gone. The final answer is that the amplitude is the sum of all possible different paths. That’s all the path integral is calculating.

There’s something very similar classically, that you might remember from your E&M class, called Huygens’s principle. It says that to calculate the propagation of electromagnetic waves, you can treat each point in the wavefront as the center of a fresh disturbance and a new source for the waves. This is useful, for example, in thinking about diffraction, where you can propagate the plane wave along to the slits, and then start the waves propagating anew from each slit. It’s totally intuitive and works for any waves. Just think of a pond with some sticks in it where you can really see the waves moving along. If the wave goes through a hole, a new wave starts from the hole and keeps going. So the path integral is doing just this kind of wave-propagation analysis, but for quantum-mechanical particle-waves, rather than classical waves.

There are a number of amazing things about path integrals. For example, they imply that by dealing with only classical field configurations you get the quantum amplitude. This is really crazy if you think about it – these classical fields all commute, so you are also getting the non-commutativity for free somehow. Time ordering also just seems to drop out. And where are the particles? What happened to second quantization? In some ways, path integrals take the wave nature of matter to be primary, while the canonical method starts with particles.

Path integrals are in many ways simpler and more beautiful the canonical quantization, but they obscure some of the physics. Nowadays, people often just start with the path integral, using it to define the theory (even though, the integral over field configurations $D\phi$ is not exactly well-defined mathematically). The canonical procedure, where we start with free fields and add interactions, is inherently perturbative. Path integrals on the other hand, don’t mention free fields at all, so they can be used to study non-perturbative things, like lattice QCD, instantons, black holes, etc. Thus path integrals are more general, but they are equivalent to canonical quantization for perturbative calculations, and provide drastically different physical intuition.

To begin, we’ll derive equation 1, then we’ll re-derive the Feynman rules and continue discussing who to interpret the path integral.
2 Gaussian integrals

Before we begin, let’s review how to compute the Gaussian integral

\[ I = \int_{-\infty}^{\infty} dp e^{-\frac{1}{2}a p^2 + Jp} \]  

(2)

First of all, complete the square.

\[ I = \int_{-\infty}^{\infty} dp e^{-\frac{1}{2}a(p - \frac{J}{a})^2 + \frac{J^2}{2a}} \]  

(3)

Then shift \( p \to p + \frac{J}{a} \). The measure doesn’t change. So

\[ I = e^{\frac{J^2}{2a}} \int_{-\infty}^{\infty} dp e^{-\frac{1}{2}ap^2} \]  

(4)

Now use a trick:

\[ \left( \int dp e^{-\frac{1}{2}p^2} \right)^2 = \int dx \int dy e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}y^2} = 2\pi \int_0^{\infty} r dr e^{-\frac{1}{2}r^2} = \pi \int_0^{\infty} dr e^{-\frac{1}{2}r^2} = 2\pi \]  

(5)

So,

\[ \int_{-\infty}^{\infty} dp e^{-\frac{1}{2}ap^2 + Jp} = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}} \]  

(6)

We can also generalize it for many \( p_i \), so that \( ap^2 = p_i a_{ij} p_j = \vec{p} A \vec{p} \), with \( A \) a matrix. If we diagonalize \( A \) then the integral is just a product of integrals over the \( p_i \), so we just get the product of all the factors above. That is,

\[ \int_{-\infty}^{\infty} d\vec{p} e^{-\frac{1}{2}\vec{p} A \vec{p} + J \vec{p}} = \sqrt{\left( \frac{2\pi}{{\text{det}} A} \right)^n} e^{\frac{1}{2}J A^{-1} J} \]  

(7)

This formula is really useful for path integrals.

3 The Path Integral

The easiest way to derive the path integral is to start with non-relativistic quantum mechanics.

3.1 path integral in QM

In non-relativistic quantum mechanics, the Hamiltonian is

\[ H(x, t) = \frac{p^2}{2m} + V(x, t) \]  

(8)

Suppose our initial state \( |i\rangle \) is a particle at \( x_i \) at time \( t_i \) and we want to project it onto the final state \( |f\rangle \) at \( x_f \) and time \( t_f \). We find

\[ \langle f | i \rangle = \langle x_f | p | x_i \rangle \sim \langle x_f e^{-iH(t_f-t_i)} | x_i \rangle \]  

(9)

This is only heuristic because \( H(t) \) and \( H(t') \) don’t commute, so we can’t just solve the Schrödinger equation in this simple way with an exponential. However, we can solve it this way for infinitesimal time intervals. So let’s break this down into small time intervals \( \delta t \) with times \( t_j = t_i + j\delta t \) and \( t_n = t_f \).

\[ \langle f | i \rangle = \int dx_1 \cdots dx_n \langle x_f \cdots e^{-iH(x_f,t_f)\delta t} | x_n \rangle \langle x_n \cdots | x_2 \rangle \langle x_2 e^{-iH(x_2,t_2)\delta t} | x_1 \rangle \langle x_1 e^{-iH(x_1,t_1)\delta t} | x_i \rangle \]  

If we use a basis of free fields

\[ \langle p | x \rangle = e^{ipx} \]  

(10)
to give
\[ \langle x_{j+1} | e^{-iH\delta t} | x_j \rangle = \int \frac{dp}{2\pi} \langle x_{j+1} | p \rangle \langle p | e^{-\left(\frac{p^2}{2m} + V(x_j, t_j)\right)\delta t} | x_j \rangle \]
\[ = e^{-iV(x_j, t_j)\delta t} \int \frac{dp}{2\pi} e^{-\frac{p^2\delta t}{2m}} e^{-ip[x_{j+1} - x_j]} \]
\[ (11) \]

Now we can use the Gaussian integral formula we computed above.
\[ \int_{-\infty}^{\infty} dp e^{-\frac{1}{2}p^2 + Jp} = \sqrt{\frac{2\pi}{a}} e^{\frac{a}{2}J^2} \]
\[ (13) \]

For our integral, \( a = \frac{\delta t}{m}, J = -i(x_{j+1} - x_j) \) so we get:
\[ \langle x_{j+1} | e^{-iH\delta t} | x_j \rangle = Ne^{-iV(x, t)\delta t} e^{\frac{i2m\delta t(x_{j+1} - x_j)^2}{(m\delta t)^2}} \]
\[ = Ne^{i\frac{1}{2}m\delta t(x_{j+1} - x_j)} \]
\[ = Ne^{iL(x, \dot{x})\delta t} \]
\[ (14) \]

where \( N \) is the normalization, which we’ll ignore. All that happened here is that the Gaussian integral performed the Legendre transform for us, to go from \( H(x, p) \rightarrow L(x, \dot{x}) \).

So,
\[ \langle f | i \rangle = N \int dx_1 \ldots dx_n e^{iL(x_n, \dot{x})\delta t} \ldots e^{iL(x_1, \dot{x_1})\delta t} \]
\[ (17) \]
and taking the limit \( \delta t \rightarrow 0 \) we get
\[ \langle f | i \rangle = N \int Dx(t) e^{iS(x, \dot{x})} \]
\[ (18) \]
where we sum over all paths \( x(t) \) and the action is \( S = \int dt L \).

4 Path integral in QFT

The field theory derivation in field theory, is very similar, but the set of intermediate states is more complicated. Let’s start with just the vacuum matrix element
\[ \langle 0, t_f | 0, t_i \rangle \sim \langle 0 | e^{-iH(t_f - t_i)} | 0 \rangle \]
\[ (19) \]
In QM we broke this down into integrals over \( |x\rangle \langle x| \) for intermediate times. The states \( |x\rangle \) are eigenstates of the \( \dot{x} \) operator. In field theory, the equivalent of \( \dot{x} \) is \( \phi(x) \), where, in case you’ve forgotten,
\[ \phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} (a_k e^{i\vec{k}\vec{x}} + a_k^* e^{-i\vec{k}\vec{x}}) \]
\[ (20) \]
This is an infinite number of operators, one at each point \( x \). We put the hat on \( \phi \) to remind you that it’s an operator. Then the equivalent of \( |x\rangle \langle x| \) is a complete set of eigenstates of \( \hat{\phi} \)
\[ \hat{\phi}(\vec{x}) |\Phi\rangle = \Phi(\vec{x}) |\Phi\rangle \]
\[ (21) \]
The eigenvalues are functions of space \( \Phi(\vec{x}) \). Of course, \( \Phi(\vec{x}) \) will have to satisfy some kind of normalization condition, so that it has a fixed number of particles, but we will worry about that later.

Now our basis is over states \( |\Phi\rangle \). So our matrix element can be chopped up into
\[ \langle 0, t_f | 0, t_i \rangle = \int \mathcal{D} \Phi_1(x) \ldots \mathcal{D} \Phi_n(x) \langle 0 | e^{-iH(t_n)\delta t} | \Phi_n \rangle \langle \ldots | \Phi_2 | e^{-iH(t_2)\delta t} | \Phi_1 \rangle \langle \Phi_1 | e^{-iH(t_1)\delta t} | 0 \rangle \]
Now each of these pieces becomes
\[\langle \Phi_{j+1} | e^{iH(t_j)\delta t} | \Phi_j \rangle = e^{i \int d^3x \mathcal{L} \Phi \Phi^\dagger \delta t} \]
And taking \( \delta t \to 0 \) gives
\[\langle 0, t_f | 0, t_i \rangle = \int \mathcal{D} \Phi(x, t) e^{i S[\Phi]} \]
where \( S[\Phi] = \int d^4x \mathcal{L} \).

So the path integral tells us to integrate over all \emph{classical} field configurations \( \Phi \). Note that \( \Phi \) is not just the one-particle states, it can have 2-particle states, etc. We can remember this by drawing pictures for the paths – including disconnected bubbles. Actually, we really sum over all kinds of discontinuous, disconnected random fluctuations, but the ones that dominate the path integral will have a nice physical interpretation as we can see.

5 Classical limit

Let’s put back \( \hbar \) for a moment. Since \( \hbar \) has dimensions of action, it is simply
\[\langle 0, t_f | 0, t_i \rangle = \int \mathcal{D} \Phi(x, t) e^{\frac{i}{\hbar} S[\Phi]} \]
(24)
Let’s forget about the \( i \) for a moment and suppose we need to calculate
\[\int \mathcal{D} \Phi(x, t) e^{-\frac{1}{\hbar} S[\Phi]} \]
(25)
In this case, the integral would clearly be dominated by \( \Phi_0 \) which is where \( S[\Phi] \) has a minimum – everything else would give a bigger \( S[\Phi] \) and be infinitely more suppressed as \( \hbar \to 0 \).

Now, when we put the \( i \) back in, the same thing happens, not because the non-minimal terms are zero, but because away from the minimum you have to sum over phases swirling around infinitely fast. When you sum infinitely swirling phases, you also get something which goes to zero. There’s a theorem that says this happens, related to the \emph{method of stationary descent} or sometimes \emph{method of steepest descent}. Another way to see it is to use the more intuitive case with \( e^{-S[\Phi]/\hbar} \). Since we expect the answer to be well defined, it should be an analytic function of \( \Phi_0 \). So we can take \( \hbar \to 0 \) in the imaginary direction, showing that the integral is still dominated by \( S[\Phi_0] \).

In any case, the classical limit is dominated by a field configuration which minimizes the action. This is the classical path that a particle would take. So the path integral has a nice intuitive classical correspondence.

6 Time-ordered products

Suppose we insert a field at fixed \( x \) and \( t \) into the path integral
\[\mathcal{I} = \int \mathcal{D} \Phi(x, t) e^{i S[\Phi]} \Phi(x_j, t_j) \]
(26)
What does this represent?
Going back through our derivation, we have
\[\mathcal{I} = \int \mathcal{D} \Phi_1(x) \cdots \mathcal{D} \Phi_n(x) \left( \Phi_1 | e^{i H(t_n)\delta t} | \Phi_n \right) \cdots \left( \Phi_2 | e^{i H(t_2)\delta t} | \Phi_1 \right) \left( \Phi_1 | e^{i H(t_1)\delta t} | 0 \right) \Phi_j(x_j) \]
since the subscript on \( \Phi \) is just it’s point in time, we know which \( \Phi_i \’s \) these correspond to. Let’s take the part with just \( \Phi_j \)
\[\int \mathcal{D} \Phi_j(x) \left\{ e^{i H(t_n)\delta t} | \Phi_j \rangle \Phi_j(x_j) \langle \Phi_j | \right\} = \phi(x_j) \int \mathcal{D} \Phi_j(x) | \Phi_j \rangle \langle \Phi_j | \]
(27)
So we get to replace \( \Phi(x_j) \) by the operator \( \hat{\phi}(x_j) \) stuck in at the time \( t_j \). Then we can collapse up all the integrals to give

\[
\int D\phi(x,t)e^{iS[\Phi]}\Phi(x_j,t_j) = \left\langle 0 \mid \hat{\phi}(x_j,t_j) \mid 0 \right\rangle
\]

(28)

If you find the collapsing-up-the-integrals confusing, just think about the derivation backwards. An insertion of \( \hat{\phi}(x_j,t_j) \) will end up by \( |\Phi_j\rangle \langle \Phi_j| \), and then pull out a factor of \( \Phi(x_j,t_j) \).

Next, observe that

\[
\Phi(x_j,t_j) e^{iS[\Phi]} \Phi(x_k,t_k) = \left\langle 0 \mid T \left\{ \hat{\phi}(x_j,t_j) \hat{\phi}(x_k,t_k) \right\} \right\rangle |0\rangle
\]

(30)

We get time ordering for free in the path integral. In general

\[
\int D\phi(x)e^{iS[\Phi]}\Phi(x_1)\cdots\Phi(x_n) = \left\langle 0 \mid T \left\{ \hat{\phi}(x_1)\cdots\hat{\phi}(x_n) \right\} \right\rangle |0\rangle
\]

(31)

Why does this work? There are a few cross checks you can do. As an easy one, suppose the answer were

\[
\int D\phi(x)e^{iS[\Phi]}\Phi(x_1)\Phi(x_2) = \left\langle 0 \mid \hat{\phi}(x_1)\hat{\phi}(x_2) \right\rangle |0\rangle
\]

(32)

Well, the left hand side doesn’t care whether I write \( \Phi(x_1)\Phi(x_2) \) or \( \Phi(x_2)\Phi(x_1) \), since these are classical fields. So what would determine what order I write the fields on the right? We see it must be something that makes the fields effectively commute, like the time-ordering operator.

From now on, we’ll just use \( \hat{\phi}(x) \) instead of \( \Phi(x) \), for the classical fields.

### 6.1 Current shorthand

There’s a great way to calculate path integrals using currents. Let’s add a source term to our action. So define

\[
Z[J] = \int D\phi \exp \left\{ iS[\phi] + i \int d^4x J(x) \phi(x) \right\}
\]

(33)

Then,

\[
Z[0] = \int D\phi e^{ij d^4x \mathcal{L}[\phi]} = |0\rangle |0\rangle
\]

(34)

Next, observe that

\[
\frac{d}{dJ(x_1)} \int d^4x J(x) \phi(x) = \phi(x_1)
\]

(35)

So,

\[
- i \frac{dZ}{dJ(x_1)} = \int D\phi \exp \left\{ iS[\phi] + i \int d^4x J(x) \phi(x) \right\} \phi(x_1)
\]

(36)

And thus

\[
- i \frac{dZ}{dJ(x_1)} \bigg|_{J=0} = \int D\phi \exp \{ iS[\phi] \} \phi(x_1) = \langle 0 | \phi(x_1) | 0 \rangle
\]

(37)

Similarly,

\[
( - i )^n \frac{d^n Z}{dJ(x_1) \cdots dJ(x_n)} \bigg|_{J=0} = \langle 0 | T \{ \phi(x_1) \cdots \phi(x_n) \} | 0 \rangle
\]

(38)

So this is a nice way of calculating time-ordered products.
7 Solving the free Path Integral

First, let’s look at the free theory.

\[ \mathcal{L} = -\frac{1}{2} \phi(\Box + m^2) \phi \]  \hfill (39)

Then

\[ Z_0[J] = \int \mathcal{D}\phi \exp \left\{ i \int d^4x \left( -\frac{1}{2} \phi(\Box + m^2) \phi + J(x) \phi(x) \right) \right\} \]  \hfill (40)

Now we can solve this exactly, since we already have our relation

\[ \int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2} \phi A\phi + J\phi} = \sqrt{\frac{(2\pi)^n}{\det A}} e^{\frac{1}{2} J A^{-1} J} \]  \hfill (41)

The path integral is just an infinite number of \( p_i \) components.

This is exactly what we have, with \( A = (\Box + m^2) \). We have already studied the inverse of \( (\Box + m^2) \),

\[ (\Box + m^2) \phi(x) = J(x) \Rightarrow \phi(x) = \int d^4x \Pi(x - y)J(y) \]  \hfill (42)

where \( \Pi \) is the Green’s function satisfying

\[ (\Box + m^2) \Pi(x - y) = \delta(x - y) \]  \hfill (43)

Explicitly,

\[ \Pi(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2} e^{ip(x - y)} \]  \hfill (44)

up to boundary conditions. Thus

\[ Z_0[J] = N \exp \left\{ i \int d^4x \int d^4y \frac{1}{2} J(x)\Pi(x - y)J(y) \right\} \]  \hfill (45)

And so,

\[ \langle 0 | T \left\{ \phi(x)\phi(y) \right\} | 0 \rangle = (-i)^2 \frac{d^2Z}{dJ(x)dJ(y)} \big|_{J=0} = i\Pi(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{ip(x - y)} \]  \hfill (46)

What happened to the time-ordering?

The standard answer is that we need to put a convergence factor in the path integral

\[ Z_0[J] = \int \mathcal{D}\phi \exp \left\{ i \int d^4x \left( -\frac{1}{2} \phi(\Box + m^2) \phi + J(x) \phi(x) \right) \right\} \exp \left\{ -i \int d^4x \phi^2 \right\} \]  \hfill (47)

\[ = \int \mathcal{D}\phi \exp \left\{ i \int d^4x \frac{1}{2} \phi(-\Box - m^2 + i\epsilon) \phi + J(x) \phi(x) \right\} \]  \hfill (48)

\[ = \exp \left\{ -\frac{1}{2} \int d^4x \int d^4y J(x)D_F(x - y)J(y) \right\} \]  \hfill (49)

Where \( D_F(x - y) \) is the Feynman propagator

\[ \langle 0 | T \left\{ \phi(x)\phi(y) \right\} | 0 \rangle = (-i)^2 \frac{d^2Z_0}{dJ(x)dJ(y)} \big|_{J=0} = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{ip(x - y)} = D_F(x - y) \]  \hfill (50)

I’m not sure how legitimate this is. It’s really hard to define the path integral mathematically, but at least this convergence factor sounds believable. In any case, we knew we had to get the time-ordered product out, so this is the correct answer.

So, for for the free field case we have solved the free path integral exactly. We get the same thing as just using the classical equations of motion and plugging back in. That is what we have been calling integrating out a field. But note that we are not integrating out the field when we do the Gaussian integral, we are actually computing the path integral. This only works for free fields, where we can solve it exactly. So there is no difference between a classical and quantum system of free fields. If there are interactions, we can approximate the integral by expanding around the minimum and then integrating out the field. This works if the quantum corrections are small, or the interactions are weak.
7.1 4-point function

We can also compute higher order products

\[
\langle 0 | T \{ \hat{\phi}(x_1) \hat{\phi}(x_2) \hat{\phi}(x_3) \hat{\phi}(x_4) \} | 0 \rangle = (-i)^4 \frac{d^4 Z_0}{d J(x_1) \cdots d J(x_4)} \bigg|_{J=0} \tag{51}
\]

\[
= \frac{d^4}{d J(x_1) \cdots d J(x_4)} e^{-\frac{i}{2} \int d^4 x d^4 y J(x) D_F(x-y) J(y)} \bigg|_{J=0} \tag{52}
\]

\[
= \frac{d^4}{d J(x_1) d J(x_2) d J(x_3)} \left( - \int d^4 z D_F(x_4 - z) J(z) \right) e^{-\frac{i}{2} \int d^4 x d^4 y J(x) D_F(x-y) J(y)} \bigg|_{J=0} \tag{53}
\]

Before we continue, let’s simplify the notation

Let us abbreviate this as

\[
\langle 0 | T \{ \hat{\phi}_1 \hat{\phi}_2 \hat{\phi}_3 \hat{\phi}_4 \} | 0 \rangle = \frac{d^4}{d J_1 d J_2 d J_3 d J_4} e^{-\frac{i}{2} J_1 J_2 J_3 J_4} \bigg|_{J=0} \tag{54}
\]

\[
= \frac{d^3}{d J_1 d J_2 d J_3} ( - J_2 D_{z4}) e^{-\frac{i}{2} J_1 J_2 J_3 J_4} \bigg|_{J=0} \tag{55}
\]

\[
= \frac{d^2}{d J_1 d J_2} ( - D_{34} + J_2 D_{z3} J_4 D_{w4}) e^{-\frac{i}{2} J_1 J_2 J_3 J_4} \bigg|_{J=0} \tag{56}
\]

\[
= \frac{d}{d J_1} ( D_{34} J_2 D_{z2} + D_{23} J_2 J_4 D_{w4} + J_2 D_{z3} D_{w4} - J_2 D_{z3} J_4 D_{w4} J_2 + J_2 J_4 J_2) e^{-\frac{i}{2} J_1 J_2 J_3 J_4} \bigg|_{J=0} \tag{57}
\]

\[
= D_{34} D_{12} + D_{23} D_{14} + D_{13} D_{24} \tag{58}
\]

So we get

\[
\langle 0 | T \{ \hat{\phi}(x_1) \hat{\phi}(x_2) \hat{\phi}(x_3) \hat{\phi}(x_4) \} | 0 \rangle = \tag{59}
\]

\[
D_F(x_3 - x_4) D_F(x_1 - x_2) + D_F(x_2 - x_3) D_F(x_1 - x_3) + D_F(x_1 - x_3) D_F(x_2 - x_4) \tag{60}
\]

These are the 3 possible contractions. This is exactly what we found from time-dependent perturbation theory.

8 Interactions

Now suppose we have interactions

\[
\mathcal{L} = - \frac{1}{2} \phi (\square + m^2) \phi + \lambda \phi^4 \tag{61}
\]

Then, we can write

\[
Z[J] = \int D\phi e^{\frac{i}{2} \int d^4 x \left[ \frac{1}{2} \phi (-\square - m^2 + i \epsilon) \phi + J(x) \phi(x) + \lambda \phi^4 \right]} \tag{62}
\]

\[
= \int D\phi e^{\frac{i}{2} \int d^4 x \left[ \phi \phi(x) + J(x) \phi(x) + \lambda \phi^4 \right] e^{i \int d^4 x \frac{\lambda}{2} \phi^4}} \tag{63}
\]

Then

\[
\langle 0 | T \{ \hat{\phi}(x_1) \hat{\phi}(x_2) \} | 0 \rangle = \langle 0 | T \{ \hat{\phi}_1(x_1) \hat{\phi}_1(x_2) \} | 0 \rangle + i \lambda \int d^4 z \langle 0 | T \{ \hat{\phi}_1(x_1) \hat{\phi}_1(x_2) \hat{\phi}_1(z) \hat{\phi}_1(z) \} | 0 \rangle + \ldots \tag{64}
\]

\[
= \langle 0 | T \{ \hat{\phi}_1(x_1) \hat{\phi}_1(x_2) e^{i \int d^4 z \lambda \hat{\phi}_1(z) \hat{\phi}_1(z)} \} | 0 \rangle \tag{64}
\]
I have added the \( I \) subscript to remind us that these are fields in the free theory, evolving with the free Hamiltonian (what we called the interaction pictur e before). So we have reproduced exactly the expression for the perturbation expansion the generated the Feynman rules. That is, have reproduced the Feynman rules from the path integral.

Another occasionally useful notation is

\[
Z[J] = e^{i \int d^4 x V[\frac{d}{dx}(x)]} \int D\phi e^{-\frac{i}{2} \int d^4 x J(x) D_x x (x) J(x)} = e^{i \int d^4 x V[\frac{d}{dx}(x)]} Z_0[J] \bigg|_{J=0}
\]

(65)

This emphasizes that everything is determined by \( Z_0 \).

Finally, remember that what we were interested in for \( S \)-matrix elements was

\[
\langle \Omega | T \{ \phi(x_1) \cdots \phi(x_n) \} | \Omega \rangle \equiv \frac{\langle 0 | T \{ \phi(x_1) \cdots \phi(x_n) e^{i \int d^4 z \lambda \phi_i(z)} | 0 \rangle}{\langle 0 | e^{i \int d^4 z \lambda \phi_i(z)} | 0 \rangle} \equiv \frac{1}{Z[J]} \bigg|_J=0
\]

(67)

Since the same \( Z \) appears in the numerator and the denominator, the normaliza tion of \( Z \) drops out, which is why we have been able to ignore it so far.

# 9 The ward identity

One of the key things that makes path integrals useful is that we can do field redefinitions. For example, say we have the path integral for some gauge theory

\[
\mathcal{L}[A, \phi] = -\frac{1}{4} F_{\mu \nu}^2 + |D_\mu \phi_i|^2 - m^2 \phi_i^2 + \cdots
\]

(69)

Then the path integral is

\[
Z[0] = \int D\phi D\phi^* e^{i \int d^4 x \mathcal{L}[A, \phi]}
\]

(70)

It is just the integral over all the fields.

Now suppose we change variables

\[
A_\mu = A'_\mu(x) + V_\mu(x)
\]

\[
\phi_i = \phi'_i(x) + \Delta_i(x)
\]

(71)

(72)

Since we are integrating over all fields, the path integral is automatically invariant, for any \( V_\mu(x) \) and any \( \Delta_i(x) \).

\[
\int D\phi D\phi^* e^{i \int d^4 x \mathcal{L}[A, \phi]} = \int D\phi D\phi^* e^{i \int d^4 x \mathcal{L}[A, \phi] + \int d^4 x \mathcal{L}[A', \phi']}
\]

(73)

This is not very impressive.

Now consider a change of variables which is a gauge transformation

\[
A_\mu(x) = A'_\mu(x) + \partial_\mu \alpha(x)
\]

\[
\phi_i(x) = e^{i \alpha(x)} \phi'_i(x)
\]

(74)

(75)

For some function \( \alpha(x) \). Then we can make stronger statements. The Lagrangian by itself is invariant, because this is a gauge transformation.

\[
e^{i \int d^4 x \mathcal{L}[A', \phi']} = e^{i \int d^4 x \mathcal{L}[A, \phi]}
\]

(76)
The measure $DA$ is invariant because this is just a linear shift. The measure $D\phi_i$ changes by a factor $e^{i\alpha(x)}$ but $D\phi_i^*$ changes by $e^{-i\alpha(x)}$ and these two factors cancel. Since for every field we need to integrate over it and its conjugate, the cancellation will always occur. So

$$\int D\phi_i^* D\phi_i^* DA_\mu = \int D\phi_i D\phi_i^* DA_\mu$$  \hspace{1cm} (77)

Still not very impressive.

But now consider the correlation function of something involving an $A_\mu$ field

$$\langle 0 | A_\mu(x_1) | 0 \rangle = \int D\phi_i^* D\phi_i^* DA_\mu e^{i\int d^4x L[A,\phi]} A_\mu(x_1)$$  \hspace{1cm} (78)

Now make the same replacement. Then any change of variables will not affect the path integral,

$$\int D\phi_i^* D\phi_i^* DA_\mu e^{i\int d^4x L[A,\phi]} A_\mu(x_1) = \int D\phi_i^* D\phi_i^* DA_\mu^* e^{i\int d^4x L[A^*,\phi^*]} A_\mu^*(x)$$  \hspace{1cm} (79)

But also, since the measure and action are separately invariant, this simplifies to

$$\int D\phi_i^* D\phi_i^* DA_\mu^* e^{i\int d^4x L[A,\phi]} A_\mu^*(x_1) = 0$$

which means

$$\int D\phi_i^* D\phi_i^* DA_\mu e^{i\int d^4x L[A,\phi]} A_\mu^*(x_1) = 0$$  \hspace{1cm} (80)

This is an example of the Ward identity.

It follows more generally as well. Suppose we add in some more fields

$$\langle 0 | A_\mu(x_1) \phi_i(x_2) \cdots \phi_j(x_n) | 0 \rangle = \int D\phi_i^* D\phi_i^* DA_\mu e^{i\int d^4x L[A,\phi]} A_\mu(x_1) \phi_i(x_2) \cdots \phi_j(x_n)$$  \hspace{1cm} (82)

$$= \int D\phi_i^* D\phi_i^* DA_\mu e^{i\int d^4x L[A,\phi]} [A_\mu(x_1) + \partial_\mu \alpha(x_1)] \phi_i(x_2) \cdots e^{i\alpha(x_n)} \phi_j(x_n)$$  \hspace{1cm} (83)

These phase factors are just numbers, which don’t matter. For example, just taking the absolute value, we find

$$|\langle 0 | A_\mu(x_1) \phi_i(x_2) \cdots \phi_j(x_n) | 0 \rangle| = |\langle 0 | A_\mu(x_1) \phi_i(x_2) \cdots \phi_j(x_n) | 0 \rangle + \langle 0 | \partial_\mu \alpha(x_1) \phi_i(x_2) \cdots \phi_j(x_n) | 0 \rangle|$$

This implies that $\langle 0 | \partial_\mu \alpha(x_1) \phi_i(x_2) \cdots \phi_j(x_n) | 0 \rangle = 0$ which is the ward identity with one $A_\mu$.

Explicitly, if we write

$$\langle 0 | A(x_1) \phi_i(x_2) \cdots \phi_j(x_n) | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} \epsilon_\mu(k) M^\mu(k, x_2, \cdots x_n) e^{ikx}$$  \hspace{1cm} (84)

we have shown

$$\int \frac{d^4k}{(2\pi)^4} \partial_\mu \alpha(x_1) M^\mu(k, x_2, \cdots x_n) e^{ikx} = -\alpha(x_1) \int \frac{d^4k}{(2\pi)^4} k_\mu M^\mu(k, x_2, \cdots x_n) e^{ikx} = 0$$  \hspace{1cm} (85)

$$\Rightarrow \int k_\mu M^\mu(k, x_2, \cdots x_n) = 0$$  \hspace{1cm} (86)

which is the Ward identity in momentum space.

Finally, suppose there are lots of $A_\mu$’s. We find

$$\langle 0 | A_\mu(x_1) \cdots A_\mu(x_n) | 0 \rangle = \langle 0 | (A_\mu(x_1) + \partial_\mu \alpha(x_1)) \cdots (A_\mu(x_n) + \partial_\mu \alpha(x_n)) | 0 \rangle$$  \hspace{1cm} (87)

For $\alpha$ infinitesimal, this implies

$$\langle 0 | \partial_\mu(x_1) \cdots A_\mu(x_n) | 0 \rangle + \langle 0 | A_\mu(x_1) \partial_\mu(x_2) \cdots | 0 \rangle + \langle 0 | A_\mu \cdots \partial_\mu \alpha(x_n) | 0 \rangle = 0$$  \hspace{1cm} (88)

which is the statement that we have to sum over all possible ways that a photon can be replaced with a $k_\mu$. Because the photons are identical particles, we would have to sum over all these graphs to get the amplitude for the polarization $\epsilon_\mu = k_\mu$, so this is exactly what we would calculate.
Thus, the path integral makes the derivation of the Ward identity easy. It is also completely non-perturbative.

10 Gauge invariance: Faddeev-Popov

Another thing that’s easy to do with path integrals is to prove gauge invariance, meaning $\xi$ dependence of the amplitudes.

Recall that vector field Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} = \frac{1}{2} A_{\mu} (-k^2 \eta_{\mu\nu} + k_{\mu} k_{\nu}) A_{\nu} + J_{\mu} A_{\mu} \quad (89)$$

And the equations of motion are

$$(-k^2 \eta_{\mu\nu} + k_{\mu} k_{\nu}) A_{\nu} = J_{\mu} \quad (90)$$

which is not invertible because this matrix has 0 determinant (it has an eigenvector $k_{\mu}$ with eigenvalue 0). The physical reason it’s not invertible is because we can’t uniquely solve for $A_{\mu}$ in terms of $J_{\mu}$ because of gauge invariance:

$$A_{\mu} \rightarrow A_{\mu} + \partial_{\mu} \alpha(x) \quad (91)$$

That is, many currents correspond to the same vector field. Our previous solution was to gauge fix by adding the term $\frac{1}{\xi} (\partial_{\mu} A_{\mu})^2$ to the Lagrangian. Now we will justify that prescription, and prove gauge invariance in general, through the Faddeev-Popov procedure.

With a general set of fields $\phi_i$ and interactions we are interested in computing

$$Z_{O} = \int D A_{\mu} D \phi_i e^{i \int d^4 x L[A, \phi_i]} O \quad (92)$$

where $O$ refers to whatever we’re taking the correlation function of (for example, $O = \phi(x)\phi(y)$)

$$\langle \Omega | T \{O\} | \Omega \rangle = \frac{Z_{O}}{Z_1} \quad (93)$$

Recall that under a gauge transformation $\partial_{\mu} A_{\mu} = \Box \alpha$. Since we can always go to a gauge where $\partial_{\mu} A_{\mu} = 0$, this means we can find a function $\alpha = \frac{1}{\Box} \partial_{\mu} A_{\mu}$. Now consider the following function

$$f(\xi) = \int D \pi e^{i \int d^4 x \frac{1}{\xi (\Box)^2}} \quad (94)$$

This is just some function of $\xi$, probably infinite. Now let’s shift

$$\pi(x) \rightarrow \pi(x) - \alpha(x) = \pi(x) - \frac{1}{\Box} \partial_{\mu} A_{\mu} \quad (95)$$

This is just a shift, so the integration measure doesn’t change. Then

$$f(\xi) = \int D \pi e^{i \int d^4 x \frac{1}{\xi (\Box - \partial_{\mu} A_{\mu})^2}} \quad (96)$$

This is still just the same function of $\xi$, which despite appearances, is independent of $A_{\mu}$. So,

$$Z_{O} = \frac{1}{f(\xi)} \int D \pi D A_{\mu} D \phi_i e^{i \int d^4 x L[A, \phi_i] + \frac{1}{\xi (\Box - \partial_{\mu} A_{\mu})^2}} O \quad (97)$$

Now let’s do the gauge transformation shift, with $\pi(x)$ as our gauge parameter.

$$A_{\mu} = A'_{\mu} + \partial_{\mu} \pi$$

$$\phi_i = e^{i \pi \phi_i} \quad (99)$$

Again the measure $D \pi D A_{\mu} D \phi_i$ and the action $L[A, \phi_i]$ don’t change. $O$ may change. For example, if it’s

$$O = A_{\mu}(x_1) \ldots A_{\mu}(x_n) \phi_i(x_{n+1}) \ldots \phi_j(x_m)$$
The $A_\mu$ transform to $A_\mu + k_\mu$ and the $k_\mu$ parts vanish by the Ward identity. The fields $\phi_i$ will give phases $e^{i\pi(x_i)}$, which don’t depend on $\xi$. Then we get

$$\int D\pi e^{i\pi(x_{n+1})\ldots e^{i\pi(x_m)}} = \int D\pi'$$

(100)

where $\pi'(x) = e^{i\pi(x_{n+1})\ldots e^{i\pi(x_m)}}\pi(x)$. So this factor is actually independent of all the $x$’s. Therefore, overall, we get some constant normalization times the gauge-fixed path integral

$$Z_0 = N_\xi \int DA_\mu D\phi_i e^{i\int d^4x L[A,\phi_i] + \frac{1}{\xi}(\partial_\mu A_\mu)^2} O$$

(101)

where $N_\xi = \frac{1}{f(\xi)}$. But since we are always computing

$$\langle \Omega |T \{O\}|\Omega \rangle = \frac{Z_O}{Z_1}$$

(102)

The $N_\xi$ drops out. The point is that $Z_O$ only depends on $\xi$ through it’s normalization. Thus the time-ordered products are $\xi$ independent.

Thus all $S$-matrix elements of gauge invariant operators are guaranteed to be independent of $\xi$, which is the statement of gauge invariance. This proof is completely non-perturbative.

### 11 Fermionic path integral

For a path integral over fermions, it’s basically the same, but we have to allow for the fact that the fermions anticommute. In the quantum theory, we saw that the quantum fields must anticommute to have a Lorentz-invariant $S$-matrix. But even classically, a Lorentz invariant spinor inner product must be antisymmetric. For example, the only Lorentz invariant constructed from a single Weyl spinor is

$$\psi_L^T \sigma^2 \psi_L = \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \left( \begin{array}{c} i \\ -i \end{array} \right) = i(\psi_1 \psi_2 - \psi_2 \psi_1) = |\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle$$

(103)

From this we concluded that $\{\psi_1, \psi_2\} = 0$ which means $\psi(x)$ is valued in the anti-commuting Grassman algebra.

The Grassman numbers is simply a set of objects which add commutatively and can by multiplied by real numbers. Consider a general function $f(\theta)$ of Grassman numbers $\theta$. Since $\theta^2 = \{\theta, \theta\} = 0$, the Taylor expansion of $f(\theta)$ around $\theta = 0$ stops at first order

$$f(\theta) = a + b\theta$$

(104)

The basic thing we want to do with these functions is integrate them over all Grassman numbers to get a real number out. Since we’re making up the rules for Grassman numbers as we go along, let’s just define

$$\int d\theta (a + b\theta) = b$$

(105)

That is,

$$\int d\theta = 0, \quad \int d\theta = 1$$

(106)

We also need a convention for multiple integrals, so we say

$$\int d\theta_1 d\theta_2 d\theta_3 = \int d\theta_1 = 1$$

(107)

Thus,

$$\int d\theta_2 d\theta_1 d\theta_1 = -1$$

(108)

And so on. These rules are internally consistent, and have some proper mathematical justification, which is totally uninteresting.
Now, let’s try our Gaussian integral

\[ I = \int d\theta e^{\frac{1}{2} \theta^2} = \int d\theta = 0 \] (109)

Since \( \theta^2 = 0 \). Let’s try with two \( \theta \)'s. Let \( \psi = (\theta_1 \theta_2)^T \) and consider \( \psi^T A \psi \). Then

\[
\psi^T A \psi = (A_{12} - A_{21}) \theta_1 \theta_2 = \det(A) \theta_1 \theta_2
\] (110)

Where we have taken \( A \) antisymmetric to write it as a determinant. The symmetric part of \( A \) simply doesn’t contribute. Then this is

\[
\int d\theta_1 d\theta_2 e^{\psi^T A \psi} = \int d\theta_1 d\theta_2 (1 + \det(A) \theta_1 \theta_2) = \det(A)
\] (111)

The generalization to \( n \) variables is simply

\[
\int d\theta e^{\psi A \psi} = \det(A)
\] (112)

This is very different from the bosonic integral

\[
\int d\tilde{\psi} e^{-\frac{1}{2} A \tilde{\psi}^2} \sim \sqrt{\frac{1}{\det A}} = (\det A)^{-1/2}
\] (113)

But that’s what it is. This det factor comes up from time to time, but mostly it is just the infinite normalization of the path integral, which we can ignore in taking ratios.

To calculate the generating functional \( Z[J] \) we need a fermionic source \( \eta \). Let’s just take 2 fields \( \psi \) and \( \bar{\psi} \), and complete the square

\[
\bar{\psi} A \psi + \bar{\eta} \psi + \bar{\psi} \eta = (\bar{\psi} + \bar{\eta} A^{-1}) A (\psi + A^{-1} \eta) - \bar{\eta} A^{-1} \eta
\] (114)

Then

\[
\int d\psi d\bar{\psi} e^{\bar{\psi} A \psi + \bar{\eta} \psi + \bar{\psi} \eta} = e^{-\bar{\eta} A^{-1} \eta} \int d\psi d\bar{\psi} e^{(\bar{\psi} + \bar{\eta} A^{-1}) A (\psi + A^{-1} \eta)}
\] (115)

\[ = e^{-\bar{\eta} A^{-1} \eta} \int d\psi d\bar{\psi} e^{\bar{\psi} A \psi} = \det(A) e^{-\bar{\eta} A^{-1} \eta}
\] (116)

\[ = N e^{-\bar{\eta} A^{-1} \eta}
\] (117)

for some normalization \( N \).

In particular the two point function is

\[
\langle 0 | T \{ \bar{\psi}(0) \psi(x) \} | 0 \rangle = \frac{d^2}{d\eta(0) d\bar{\eta}(x)} \int D\psi D\bar{\psi} e^{\bar{\psi} A \psi + \bar{\eta} \psi + \bar{\psi} \eta} \big|_{\eta=0} = N A^{-1}
\] (118)

For a free Dirac field,

\[
Z[\eta] = \int D\psi D\bar{\psi} e^{\int d^4 x (\bar{\psi} i \gamma \cdot \partial - m + i \varepsilon) \psi + \bar{\eta} \psi + \bar{\psi} \eta}
\] (119)

where again, the \( i \varepsilon \) comes in to make the path integral converge. Then \( A = i \gamma \cdot \partial - m + i \varepsilon \) and we have

\[
\langle 0 | T \{ \bar{\psi}(0) \psi(x) \} | 0 \rangle = \frac{1}{Z[\eta]} \frac{d^2 Z[\eta]}{d\eta(0) d\bar{\eta}(x)} \big|_{\eta=0} = i(i \gamma \cdot \partial - m + i \varepsilon)^{-1}
\] (120)

\[
= \int \frac{d^4 p}{(2\pi)^4} e^{i p \cdot x} e^{i p \cdot x}
\] (121)

This simplifies using \((\not{p} - m)(\not{p} + m) = p^2 - m^2\), which implies

\[
\frac{1}{\not{p} - m + i \varepsilon} = \frac{\not{p} + m}{p^2 - m^2 + i \varepsilon}
\] (122)
Instead, the measure is invariant. The Lagrangian is not invariant, since we have not transformed and integrate the $\partial_\alpha \delta$ functions, up to contact interactions where $x \rightarrow x_j$, where $x_j$ is another point appearing in the correlation function. Note that this equation is exact. We derived it by expanding to first order in $\alpha$, but that just means we take infinitesimal transformations, which is not the same as calculating the correlation function to first order in $e$.

\section{12 Schwinger-Dyson equations}

An important feature of the path integral is that it tells us how the classical equations of motion are modified in the quantum theory. These are known as Schwinger-Dyson equations.

Consider the correlation function of $\bar{\psi}(x_1)\psi(x_2)$:

$$Z_{12} = \int DA_\mu D\psi D\bar{\psi} \exp \left( i \int d^4x \left[ -\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi} (i\partial_\mu + eA_\mu + m)\psi \right] \right)$$

Under a field redefinition,

$$\psi \rightarrow e^{i\alpha} \psi$$

the measure is invariant. The Lagrangian is not invariant, since we have not transformed $A_\mu$. Instead,

$$\bar{\psi}(x)\partial_\alpha \psi(x) \rightarrow \bar{\psi}(x)\partial_\alpha \psi(x) + i \partial_\mu \alpha(x) \bar{\psi}(x)\gamma^\mu \psi(x)$$

But since we are integrating over all $\psi$, the path integral is invariant. Thus, expanding to first order in $\alpha$

$$0 = \int DA_\mu D\psi D\bar{\psi} e^{iS}$$

$$\times \left[ \int d^4x i \partial_\mu \alpha(x) \bar{\psi}(x)\gamma^\mu \psi(x) \right] \bar{\psi}(x_1)\psi(x_2) - i \alpha(x_1) \bar{\psi}(x_1)\psi(x_2) + i \alpha(x_2) \bar{\psi}(x_1)\psi(x_2)$$

We can simplify this by adding $\delta$ functions: $\alpha(x_1) = \int d^4x \delta(x-x_1) \alpha(x)$. Also, we define

$$j_\mu(x) = \bar{\psi}(x)\gamma^\mu \psi(x)$$

and integrate the $\partial_\mu \alpha j^\mu$ term by parts. Then,

$$0 = \int DA_\mu D\psi D\bar{\psi} e^{iS} \int d^4x \alpha(x) \left[ \partial_\mu j^\mu(x) - \delta(x-x_1) + \delta(x-x_2) \right] \bar{\psi}(x_1)\psi(x_2)$$

Dividing by $Z[0]$, this implies

$$\partial_\mu \langle 0| T \{ j^\mu(x_1) \psi(x_2) \} \rangle = \delta(x-x_1) \langle 0| T \{ \psi(x_1)\psi(x_2) \} \rangle - \delta(x-x_2) \langle 0| T \{ \psi(x_1)\psi(x_2) \} \rangle$$

This is known as a Schwinger-Dyson equation.

\subsection{12.1 Ward identity from Schwinger-Dyson equations}

Recall that classically, $\partial_\mu j^\mu = 0$. So this says that that equation still holds within correlation functions, up to contact interactions where $x \rightarrow x_j$, where $x_j$ is another point appearing in the correlation function. Note that this equation is exact. We derived it by expanding to first order in $\alpha$, but that just means we take infinitesimal transformations, which is not the same as calculating the correlation function to first order in $e$. 
While this equation looks messy, it is actually just a generalization of the Ward identity. To see that, Fourier transform

\[ \langle 0|T\{j^\mu(x)\bar{\psi}(x_1)\psi(x_2)\}\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} e^{ipx} e^{iq_1x_1} e^{iq_2x_2} M_\mu(p, q_1, q_2) \]  

(132)

\[ \langle 0|T\{\bar{\psi}(x_1)\psi(x_2)\}\rangle = \int \frac{d^4q_2}{(2\pi)^4} \frac{d^4q_1}{(2\pi)^4} e^{iq_1x_1} e^{iq_2x_2} M_0(q_1, q_2) \]  

(133)

So,

\[ \int d^4x d^4x_1 d^4p x_1 e^{ipx} e^{iq_1x_1} e^{iq_2x_2} \delta(x - x_1) \langle 0|T\{\bar{\psi}(x_1)\psi(x_2)\}\rangle = M_0(q_1 + p, q_2) \]  

(134)

Then we have

\[ p_\mu M_\mu(p, q_1, q_2) = M_0(q_1 + p, q_2) - M_0(q_1, q_2 + p) \]  

(135)

This is known as a generalized Ward-Takahashi identity.

This equation actually implies the Ward identity. To see it, first, observe that \(M_0(q, q')\) is just the two point function for an electron propagator with incoming momentum \(q\) and outgoing momentum \(q'\). By momentum conservation, it will vanish unless \(q = q'\). Thus \(M_0(q, q') = M_0(q)\).

The Ward identity, with \(q = q_1\) incoming is then

\[ [p_\mu M_\mu(p, q) = M_0(q + p) - M_0(q)] \]  

(136)

Now if we put the electron on shell, as we must for S-matrix elements by LSZ, then \(q = m\) and \(M_0(q + p) = M_0(q)\), since neither depend on the momentum factors. So we find \(p_\mu M_\mu = 0\).

The function \(M_\mu\) is directly related to matrix elements of photons. Indeed, the interaction \(j^\mu(x)\bar{\psi}(x_1)\psi(x_2)\) term is the matrix element of \(\bar{\psi}(x_1)\psi(x_2)\) in the presence of an external current. The current interacts through the photon, so \(j^\mu\) can be thought of as a place holder for the photon polarization. So,

\[ \langle 0|T\{A_\mu(p)\bar{\psi}(q_1)\gamma^\mu\psi(q_2)\}\rangle = \varepsilon_\mu M_\mu(p, q_1, q_2) = \varepsilon_\mu M_\mu(p, q) \]  

(137)

So \(p_\mu M_\mu\) is the regular Ward identity for S-matrix elements we have been using all along. This is another way to prove it.

The more general Ward-Takahashi identity is especially useful for proving non-perturbative statements about matrix elements when the electron is not on-shell: \(M_0(q) \neq M_0(q + p)\). One example is the non-renormalization of electric charge, which we will discuss shortly.

### 13 BRST invariance

There is a beautiful symmetry called BRST invariance, which is a residual exact symmetry of the Lagrangian even after gauge fixing. It is particularly useful for studying more complicated gauge theories, but it is a little easier to understand in the QED case.

Notice that the gauge fixing term \((\partial_\mu A_\mu)^2\) is invariant under gauge transformations \(A_\mu \to A_\mu + \partial_\mu \alpha\) for \(\Box \alpha = 0\). So this is a residual symmetry of the entire Lagrangian. As we have already seen, it is awkward to deal with constrained theories, and constraints on symmetries are no different. So consider adding to the Lagrangian two new free fields \(c\) and \(d\)

\[ \mathcal{L} = \mathcal{L}[A, \phi] + \frac{1}{\xi} (\partial_\mu A_\mu)^2 - d \Box c \]  

(138)

So the path integral becomes

\[ Z_{\Box} = \int Dc Dd DA D\phi e^{i \int d^4x \mathcal{L}[A, \phi] + \frac{1}{\xi} (\partial_\mu A_\mu)^2 - d \Box c} \]  

(139)

The fields \(d\) and \(c\) are free, so their path integral can be solved exactly. The equations of motion are \(\Box c = \Box d = 0\).
This path integral has an unconstrained global symmetry, parametrized by $\theta$, for which
\begin{align}
\Delta \phi_i &= \theta c \phi_i \\
\Delta A_\mu &= \theta \partial_\mu c(x) \\
\Delta d &= \theta \frac{2}{\xi^2} (\partial_\mu A_\mu) \\
\Delta c &= 0
\end{align}
\begin{align}
(140) & \\
(141) & \\
(142) & \\
(143) & \\

We can check
\begin{align}
\Delta \left( \frac{1}{\xi} \partial_\mu A_\mu \right)^2 &= \frac{2}{\xi^2} \epsilon (\partial_\mu A_\mu) \Box c = (\Delta d) \Box c
\end{align}
(144)

All we did was replace the constrained transformation with $\Box \alpha = 0$ with an unconstrained one. But it is the same, since $\Box c = 0$ is exact because $c$ is a free field.

So we have isolated a residual symmetry in the gauge fixed Lagrangian. It is called BRST invariance. Actually, for technical reasons, BRST requires that $\theta$ be an anti-commuting Grassman number. You can check that if $\theta$ and $b$ and $c$ are fermionic, then the transformation is nilpotent $\Delta^2 = 0$. But none of this is relevant for QED.

To see why BRST is powerful, consider free solutions
\begin{align}
A_\mu(x) &= \sum_i a_i \int \frac{d^4k}{(2\pi)^4} \epsilon_\mu(k) e^{ikx} \\
\epsilon(x) &= a_c \int \frac{d^4k}{(2\pi)^4} e^{ikx} \\
d(x) &= a_d \int \frac{d^4k}{(2\pi)^4} e^{ikx}
\end{align}
(145-147)

So the free solutions are characterized by 6 numbers. For the $A_\mu$ field, there are two physical transverse modes $\epsilon_\mu^{1,2}$ which are normalizable and transverse $k_\mu \epsilon_\mu^{1,2} = 0$. The two unphysical modes are the backwards polarized mode $\epsilon_\mu^b$, which is not transverse $k_\mu \epsilon_\mu^b = 1 \neq 0$, and the forward polarized mode $\epsilon_\mu^f = k_\mu$, which is not normalizable. So we can think of free particles as being characterized by one big 6-component multiplet
\begin{align}
(a_1, a_2, a_f, a_b, a_c, a_d)
\end{align}
(148)

Under BRST transformations, $\Delta A_\mu = \theta k_\mu c = \theta a_c \epsilon_f$ so $\Delta a_f = \theta a_c$. Since $\Delta d = \theta \frac{2}{\xi^2} k_\mu A_\mu = \frac{2}{\xi^2} \theta$ so $\Delta a_d = \theta \frac{2}{\xi^2} a_b$. Thus
\begin{align}
(a_1, a_2, a_f, a_b, a_c, a_d) \rightarrow (a_1, a_2, a_f + \theta a_c, a_b, a_c, a_d + \theta \frac{2}{\xi^2} a_b)
\end{align}
(149)

The point is that the only fields involved in this transformation of the unphysical polarizations and the “ghost” fields $a$ and $b$. Since the global symmetry corresponds to a conserved current, by Noether’s theorem, only the unphysical particles are charged under the symmetry. So if we start with a state with no unphysical particles, by charge conservation, we will end up in a state with no unphysical particles. This means that the $S$-matrix restricted to physical states is unitary. We knew this already, since QED has a Ward identity, but for more complicated gauge theories, the extra symmetry is an important tool.