Unravelling in Two-Sided Matching Markets and Similarity of Preferences

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ONE-TO-ONE MATCHING (LABOR MARKET)

- firms and workers
  - each firm wants to employ at most one worker
  - each worker wants to work for at most one firm
  - each firm has preferences over workers
  - each worker has preferences over firms
THIS PAPER

- formally models unravelling with a stable clearinghouse

  - unravelling is more likely to occur when preferences are more similar

- formally investigates mechanism design problem of the optimal clearinghouses when unravelling is possible

  - a mechanism must preclude unravelling, but does not need to be stable to be Pareto-optimal
  
  - in every market there is a mechanism that achieves a Pareto-optimal outcome
OUTLINE

I. model

II. equilibria under the stable mechanism
   A. equilibria without unravelling
   B. equilibria with unravelling
   C. equilibria and similarity of preferences

III. mechanism design

IV. conclusions
I. MODEL

- $F$ firms, $f \in \{1, \ldots, F\}$
- workers have identical preferences: $F > F - 1 > \ldots > 1 > \emptyset$
- $u_f$ – utility to a worker from being matched with firm $f$;

$$u = [u_1, u_2, \ldots, u_F] \text{ s.t. } 0 < u_1 < \ldots < u_{F-1} < u_F$$

- $W$ workers, $W > F$
- firms potentially have different preferences over workers
- firm $f$’s ranking over workers: $\mathcal{R}^f = (r_{f1}^f, r_{f2}^f, \ldots, r_{fW}^f)$

$$r_{fW}^f \text{ – identity of the most preferred worker of firm } f$$

$$r_{f1}^f \text{ – identity of the least preferred worker of firm } f$$

- $v_k$ – value to $f$ of being matched with $r_k^f$

$$v = [v_1, v_2, \ldots, v_W] \text{ s.t. } 0 < v_1 < \ldots < v_{W-1} < v_W$$
FIRMS’ PREFERENCES (RANKINGS)

- $\mathcal{R}^f \in \mathcal{R}$ – set of all possible $W!$ rankings

- $\mathcal{R}^1, \ldots, \mathcal{R}^F$ drawn from joint distribution $G(\mathcal{R}^1, \ldots, \mathcal{R}^F)$

- **Independent** preferences, $G_0$: $G(\mathcal{R}^i|\mathcal{R}^{-i}) \equiv U[\mathcal{R}]

- **Identical** preferences, $G_1$: $\mathcal{R}^1 \equiv \mathcal{R}^2 \equiv \ldots \mathcal{R}^F = \mathcal{R}$, $\mathcal{R} \sim U[\mathcal{R}]

- Intermediate cases: $G_\rho = \rho G_1 + (1 - \rho)G_0$, $\rho \in (0, 1)$

$\rho$ – measure of preference similarity and comparative statics parameter

- Preferences are **more similar** under $G_\rho'$ than $G_\rho$ when $\rho' > \rho$
MATCHING

- a function from firms to workers and $\emptyset$

$$\mu: \{1, \ldots, F\} \rightarrow \{1, \ldots, W\} \cup \{\emptyset\}$$

- no two firms are matched with the same worker

STABLE MATCHING, $\mu_S$

- there is unique stable matching for any preference profile
- firm $F$ matched with its most preferred worker
- firm $F-1$ matched with its most preferred worker from the remaining
- firm $f$ matched with its most preferred worker out of those remaining after all better firms are matched
MATCHING MECHANISM

- firms are asked to report their rankings
- a matching is applied according to the reported rankings

▸ mechanism is *incentive compatible*
if no firm has incentive to misreport its preferences

STABLE MATCHING MECHANISM

▸ the stable matching is applied to the reported rankings
▸ this mechanism is incentive compatible
THE GAME

- market \((F, W, u, v, \rho, M)\)

\((F, W, u, v, \rho, M)\) and workers’ preferences commonly known

- firms simultaneously make (or not) early offers
- workers, who received an offer accept or reject it

\(t = 1\)

- matched firms and workers leave the market

\(t = 2\)

- matching mechanism \(M\) is applied to agents remaining in the market
- each firm’s ranking is realized

- no discounting
- making offers – costless
EQUILIBRIUM

- pure strategies
  - $f$'s strategy, $\sigma_f \in \{1, \ldots, W\} \cup \{\emptyset\}$
  - $\Omega_w$ – set of early offers $w$ received
  - $w$'s strategy, $\sigma_w(\Omega_w) \in \Omega_w \cup \{\emptyset\}$

- sequential equilibrium – set of strategies and beliefs:
  - strategies are sequentially rational
  - beliefs are consistent with strategies played
  - off equilibrium path – workers receive unexpected offers
    - a worker updates his belief
      - only about the firm that made him the unexpected offer
## II. EQUILIBRIA UNDER STABLE MECHANISM

- stable matching mechanism operates in $t = 2$
EARLY CONTRACTING

▶ w’s trade off:

\[ f \text{ in } t = 1 \quad \text{or} \quad \text{lottery over remaining firms in } t = 2 \]

(possibly better firm, possibly worse)

▶ f’s trade off:

average worker in \( t = 1 \) or \( \mu_S \) in \( t = 2 \)

(better firms might take f’s best workers)

to unravel a firm needs to be good enough to be accepted in \( t = 1 \) and bad enough to want to unravel

▶ in \( \mu_S \) lower ranked firms get lower expected payoff

\[ \sim \Rightarrow \text{worse firms are more eager to unravel} \]

▶ in \( \mu_S \), the higher similarity of preferences, the lower each firm’s expected payoff

\[ \sim \Rightarrow \text{under higher similarity of preferences more firms are eager to unravel} \]
A. EQUILIBRIA WITHOUT UNRAVELLING

- Suppose that no offers are made and accepted in the first period.

- It is an equilibrium when no firm that wants to contract early is accepted.

- \( \mathcal{A} \) – the set of firms whose offers would be accepted, if they were made.

- \( \mathcal{O} \) – the set of firms who would make offers, if they were accepted.

- “no unravelling” is an equilibrium when \( \mathcal{A} \cap \mathcal{O} = \emptyset \).
introduction

I. model

II. equilibria under stable mechanism

III. mechanism design

conclusions

A. equilibria without unravelling

B. equilibria with unravelling

C. equilibria and $\rho$

ACCEPTANCE SET, $\mathcal{A}$

• a worker accepts $f$’s offer in $t = 1$ when

$$u_f > \frac{1}{W} \sum_{i=1}^{F} u_i$$

expected utility in $t=2$

• observations

- the best firm is always accepted ($L \leq F$)
- it may be that all firms are accepted ($L$ may be equal to 1)
- as $W$ increases, $L$ decreases
A. equilibria without unravelling

**OFFER SET, \( \mathcal{O} \)**

- \( f \) prefers to contract in \( t = 1 \) when

\[
\frac{1}{W} \sum_{k=1}^{W} v_k > E\pi_f(\mu_S)
\]

early offer

**Observations**

- independent preferences, \( G_0 \): \( \mathcal{O} \) is empty
- intermediate cases, \( G_\rho \): as \( \rho \) increases, \( \mathcal{O} \) increases
- identical preferences, \( G_1 \): \( \mathcal{O} \) may be nonempty
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EXAMPLE FOR $G_1$

$F = 3, \ W = 4, \ v = [1, 2, 3, 4]$

- early offer yields $2.5 > 2$

- firm 1 prefers early contracting

$u = [4, 5, 6]$

- a worker's expected utility from $\mu_S$ is $\frac{15}{4} < 4$

- firm 1 is accepted in $t = 1$
INCENTIVES TO DEVIATE FROM “NO UNRAVELLING”

- workers’ acceptance set $\mathcal{A}$ – independent on $\rho$
  
  $\mathcal{A}$

- firms’ offer set $\mathcal{O}$ – increases with $\rho$

  $\mathcal{O}$

- depending on $\rho$:
  - independent, $G_0$: “no unravelling” is an equilibrium
  - identical, $G_1$: possible deviation from “no unravelling”
  - intermediate, $G_\rho$: “no unravelling” may be an equilibrium for a lower $\rho$ and not for a higher $\rho$
market does not unravel in equilibrium for low similarity of preferences but it is not necessarily true for higher similarity of preferences

Holding other market parameters constant

\[ \exists \rho^{**} \in (0, 1] \]

- for \( \rho \leq \rho^{**} \) “no unravelling” is an equilibrium
- for \( \rho > \rho^{**} \) “no unravelling” is not an equilibrium
UNRAVELLING IN EQUILIBRIUM

- set of firms that contract early – equilibrium unravelling set $\mathcal{U}^*$

- $\mathcal{U}$ is an interval, i.e. it has no ”holes”

- multiple equilibria (multiple unravelling sets) typically exist in a market

- all equilibrium unravelling sets for the same market must be “nested”

- equilibrium unravelling sets may be ordered: $\mathcal{U}^{MIN}$ and $\mathcal{U}^{MAX}$
**Proposition**

*Holding other market parameters constant, there exist $\rho^*$ and $\rho^{**}$ such that $0 < \rho^* \leq \rho^{**} \leq 1$ and*

- $\rho \in [0, \rho^*] \implies U^{MAX}(\rho) = \emptyset$
- $\rho \in (\rho^*, \rho^{**}] \implies U^{MIN}(\rho) = \emptyset \ & \ U^{MAX}(\rho) \neq \emptyset$
- $\rho \in (\rho^{**}, 1] \implies U^{MIN}(\rho) \neq \emptyset$
III. MECHANISM DESIGN

- mechanism design problem

- social planner chooses a mechanism for $t = 2$
- firms decide about early offer
- workers decide to accept or reject early offers
- all agents that did not contract in $t = 1$, participate in the mechanism in $t = 2$
- the goal of the social planner is to provide a Pareto-optimal outcome from the ex-ante perspective
an outcome, \( o \), is a function from the profile of rankings, \( R = (R^F, \ldots, R^1) \), to randomization over matchings between all firms and all workers

\[
o : \mathbf{R} \mapsto \text{Lottery}(\mu(\{1, \ldots, F\}, \{1, \ldots, W\}))
\]

outcome \( o' \) \textbf{Pareto-dominates} outcome \( o'' \) when every agent’s ex-ante expected payoff is at least the same under \( o' \) as under \( o'' \), and some agent’s ex-ante expected payoff is strictly higher under \( o' \) than under \( o'' \)

outcome is \textbf{Pareto-optimal} when there is no outcome that Pareto-dominates it

a mechanism-equilibrium pair \((M, \sigma)\) determines a unique outcome
PARETO-OPTIMAL \((\mathcal{M}, \sigma)\)

FIRST-BEST PARETO-OPTIMAL (unconstrained)

- a mechanism-equilibrium pair is first-best Pareto-optimal when it produces a Pareto-optimal outcome

SECOND-BEST PARETO-OPTIMAL (constrained)

- a mechanism-equilibrium \((\mathcal{M}, \sigma)\) pair is second-best Pareto-optimal when there is no other mechanism-equilibrium pair \((\mathcal{M}', \sigma')\) such that the outcome of \((\mathcal{M}', \sigma')\) strictly Pareto-dominates the outcome of \((\mathcal{M}, \sigma)\)
ANONYMITY

- A mechanism is **anonymous** if it assigns workers to firms based on firms’ rankings, but not on workers’ identities.
  - $\mathcal{M}_S$ is anonymous
  - A mechanism that always matches firm 1 with worker 1 is not

- Vector of strategies $\sigma$ is anonymous if any firm that contracts with a worker in $t = 1$, selects a worker at random, paying no attention to his identity

- Mechanism-equilibrium pair $(\mathcal{M}, \sigma)$ is anonymous when $\mathcal{M}$ and $\sigma$ are anonymous

- Under anonymous $(\mathcal{M}, \sigma)$ every worker’s ex-ante expected utility is the same
Proposition

For any anonymous \((\mathcal{M}, \sigma)\) which exhibits unravelling, there exists anonymous \((\mathcal{M}', \sigma')\) such that it does not exhibit unravelling and strictly Pareto-dominates \((\mathcal{M}, \sigma)\).

proof

Suppose that \((\mathcal{M}, \sigma)\) produces \(U^M \neq \emptyset\). Then, consider alternative \(\mathcal{M}'\):

1. for all \(f \in U^M\), \(\mathcal{M}'\) tentatively assigns a random worker (with probability \(\frac{1}{W}\) a firm gets its worst worker)
2. all other firms are matched according to \(\mathcal{M}\)
3. “worst workers correction”: for any firm in \(U^M\) that got its worst worker in (1), \(\mathcal{M}'\) substitutes a different unmatched worker for the worst worker
Proposition

For any anonymous $(\mathcal{M}, \sigma)$ which exhibits unravelling, there exists anonymous $(\mathcal{M}', \sigma')$ such that it does not exhibit unravelling and strictly Pareto-dominates $(\mathcal{M}, \sigma)$.

proof

Suppose that $(\mathcal{M}, \sigma)$ produces $\mathcal{U}^{\mathcal{M}} \neq \emptyset$. Then, consider alternative $\mathcal{M}'$:

1. for all $f \in \mathcal{U}^{\mathcal{M}}$, $\mathcal{M}'$ tentatively assigns a random worker (with probability $\frac{1}{W}$ a firm gets its worst worker)
2. all other firms are matched according to $\mathcal{M}$
3. “worst workers correction”: for any firm in $\mathcal{U}^{\mathcal{M}}$ that got its worst worker in (1), $\mathcal{M}'$ substitutes a different unmatched worker for the worst worker

- there exists an equilibrium without unravelling under $\mathcal{M}'$ – call it $\sigma'$
- firms in $\mathcal{U}^{\mathcal{M}}$ have higher expected payoff under $(\mathcal{M}', \sigma')$
- other firms have the same expected payoff
- all workers have the same expected utility

$(\mathcal{M}', \sigma')$ Pareto-improves $(\mathcal{M}, \sigma)$
STABLE MECHANISM

- $\mu_S$ is a Pareto-optimal outcome

- $M_S$ is anonymous

- The stable mechanism is (first- and second-best) Pareto-optimal if and only if it does not unravel

- If the stable mechanism unravels, there exists an unstable mechanism that achieves a Pareto-optimal outcome
Proposition

For every market, there exists a first-best Pareto-optimal \((\mathcal{M}, \sigma)\).

proof

Consider \(\mathcal{M}\) that

1. assigns a number from \(\{1, \ldots, F\}\) to every firm at random
2. stable algorithm with respect to the assigned numbers

- \(\mathcal{M}\) is incentive compatible
- there exists equilibrium without unravelling, \(\sigma\)
- outcome of \((\mathcal{M}, \sigma)\) is Pareto-optimal
Observation

If \((M_S, \sigma)\) exhibits unravelling, there exists a first-best Pareto-optimal \((M', \sigma')\) that strictly Pareto-improves on \((M_S, \sigma)\).

- \((M_S, \sigma)\) produces \(U \neq \emptyset\)

**consider** \(M'\):

1. Every firm \(f \in U\) draws a random number out of \(\{1, \ldots, W\}\).

2. All other firms – starting from the highest ranked – get the highest number available.

3. Firms are matched with their best available worker in order of their numbers – starting from the one with the highest number.

- \(M'\) is incentive compatible and anonymous
- there exists an equilibrium without unravelling, \(\sigma'\)
- the outcome of \((M', \sigma')\) is Pareto-optimal
- \((M', \sigma')\) strictly Pareto-improves on \((M_S, \sigma)\)
CONCLUSIONS

- unravelling with a stable clearinghouse
  - unravelling is more likely to occur as firms’ preferences grow more similar

- mechanism design when unravelling is possible
  - an anonymous mechanism must preclude unravelling to be second-best Pareto-optimal
  - there is a first-best Pareto-optimal anonymous mechanism for every market
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EQUILIBRIUM

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  - $f$’s strategy, $\sigma_f \in \{1, \ldots, W\} \cup \{\emptyset\}$
  - $\Omega_w$ – set of early offers $w$ received

- $w$’s strategy, $\sigma_w(\Omega_w) \in \Omega_w \cup \{\emptyset\}$

- sequential equilibrium – set of strategies and beliefs:
  - strategies are sequentially rational
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  - off equilibrium path – workers receive unexpected offers

  a worker updates his belief
  only about the firm that made him the unexpected offer
• $\mathcal{U}$ is an interval, i.e. it has no "holes"

Suppose, it has...

\[ f \not\in \mathcal{U}^* \]
in any market an equilibrium in pure strategies exists

- consider a market where there is no equilibrium without unravelling

\[ A \cap O \]

- the payoff of staying for \( t = 2 \) decreases \( \rightarrow \) more firms could unravel

- until it stops
MULTIPLE EQUILIBRIA

- all equilibrium unravelling sets for the same market must be “nested”

Suppose not...

- for a worker, staying for $t=2$ yields higher expected utility under $U'$ than under $U^*$

- $f \notin U^*$ only if $f$ would not be accepted (i.e., worker prefers to wait for $t = 2$)

- but if $f \in U'$, then $f$ is better than waiting for $t = 2$

- $f$ must be accepted under $U^*$
THE ROLE OF ANONYMITY

EXAMPLE

- $\mathcal{M}$ assigns firm 1 to worker 1, firm 2 to worker 2, etc.

- $\sigma$ s.t. firm $F$ contracts early with worker $F$ and all other firms wait for $\mathcal{M}$

- expected utility of worker $i$ under ($\mathcal{M}, \sigma$) is
  \[ u_i \quad \text{for} \quad i = 1, \ldots, F \]
  \[ 0 \quad \text{for} \quad i = F + 1, \ldots, W \]

- each firm’s expected payoff is
  \[ \frac{1}{W} \sum_{k=1}^{W} v_k \]

- despite unravelling, cannot be Pareto-improved
Roth and Xing (1994) on “summer associate positions” in American law firms (considered a proxy for entry-level hiring):

By the late 1970’s, even first-year students were being offered summer associate positions, and by the middle of the 1980’s the unraveling of recruiting had proceeded to such an extent that some students were being offered summer associate positions before they had matriculated at law school.

Kozinski, on why he opposes any move toward the stable clearinghouse:

... not all clerkships are created equal. (...) Prestige counts. Some circuits, the C.D. Circuit in particular, tend to draw a disproportionate share of the nation’s top applicants. Seniority matters. (...) The problem with many reform proposals is that they tend to reinforce these patterns by decreasing the means by which less-favored clerkships can compete for desirable applicants.

... such plans eliminate a very important bargaining tool for judges competing for the most gifted clerkship candidates – the ability to make offers early and entice applicants into ending the anxiety and uncertainty by accepting early.