- An equation, on the other hand, establishes an equality of two expressions; we can manipulate an equation in any way that does not alter the equality of the two sides. The governing criterion for working with equations is that balance be maintained.

  Adding/subtracting the same thing to/from both sides of the equation maintains the balance. Multiplying/dividing both sides of the equation by the same nonzero quantity maintains the balance.

  If $A = C$, where A and C are both positive, then $A^n = C^n$. For example, squaring both sides of an equation ($n = 2$) maintains balance (although it may introduce extraneous roots). Similarly, taking reciprocals of both sides of an equation ($n = -1$) maintains balance. If $J + K = C$, then $(J + K)^n = C^n$. In particular, if $J + K = C$, then $\frac{1}{J+K} = \frac{1}{C}$.

  If $A = C$, where A and C are both positive, then $\ln A = \ln C$. For example, if $J + K = C$, then $\ln(J + K) = \ln C$.

  If $A = C$, then $b^A = b^C$ for any positive constant b. For example, if $J + K = C$ then $e^{J+K} = e^C$.

We have presented two different successful approaches to tackling the problem presented in Example 17.1(b). The first dealt with the *expression* x^{x+1}; we used the fact that $e^{\ln A} = A$ for any positive A to convert the *expression* x^{x+1} to the *equivalent expression* $e^{(x+1)\ln x}$. The second approach dealt with the *equation* $f(x) = x^{x+1}$. We took the natural logarithm of both sides of the *equation* to obtain an *equivalent equation*, differentiated both sides of the *equation*, and solved for $f'(x)$. This latter technique of differentiation is called **logarithmic differentiation**.

PROBLEMS FOR SECTION 17.1

Differentiate the following.

1. (a) $y = 3^x$ (b) $y = x^3$ (c) $y = x^x$, where $x > 0$.
2. $y = (x + 1)^{(x+1)}$, where $x > -1$.
3. $y = (3x^2 + 2)^x$
4. $y = x^{x^2}$, where $x > 0$.

17.2 LOGARITHMIC DIFFERENTIATION

Logarithmic differentiation deals with the task of differentiating a positive function $f(x)$ by working with both sides of the equation $y = f(x)$ as follows.

Using Logarithmic Differentiation to Find y'

1. Begin with an equation $y = f(x)$, where $f(x) > 0$. Take the natural logarithm of *both sides* of the equation.

2. Use log rules to bring down exponents and/or simplify expressions.
3. Differentiate both sides of the equation. y is a function of x so the Chain Rule *must* be applied to differentiate $\ln y$ or $\ln f(x)$.

$$\frac{d}{dx}[\ln f(x)] = \frac{1}{f(x)} f'(x) \text{ or, equivalently, } \frac{d}{dx}[\ln y] = \frac{1}{y}\frac{dy}{dx}.$$

4. Solve for $\frac{dy}{dx}$ or $f'(x)$.
5. To express y' in terms of x, replace y or $f(x)$ with the equivalent expression in terms of x.

When to Use Logarithmic Differentiation

i. The technique is useful in differentiating a function that has the variable in both the base and the exponent.

ii. We may choose to use logarithmic differentiation to make the differentiation of quotients or products more palatable, provided that we only take the log of positive quantities.

EXAMPLE 17.2 Let $y = 2x^{e^x}$, where $x > 0$. Find $\frac{dy}{dx}$.

SOLUTION The variable is in the base and the exponent; logarithmic differentiation enables us to bring down the expression in the exponent.

$y = 2x^{e^x}$	
$\ln y = \ln\left(2x^{e^x}\right)$	Take the natural logarithm of each side.
$\ln y = \ln 2 + e^x \ln x$	Use log rules (i) and (iii) to bring down the exponent.
$\frac{d}{dx}[\ln y] = \frac{d}{dx}[\ln 2 + e^x \ln x]$	Differentiate each side. ln 2 is a constant.
$\frac{1}{y}\frac{dy}{dx} = (e^x)(\ln x) + (e^x)\left(\frac{1}{x}\right)$	Use the Chain Rule on left because y is a function of x.
$\frac{dy}{dx} = e^x\left(\ln x + \frac{1}{x}\right) y$	Solve for $\frac{dy}{dx}$.
$\frac{dy}{dx} = e^x\left(\ln x + \frac{1}{x}\right)\left(2x^{e^x}\right)$	Replace y by its expression in x. ◆

EXAMPLE 17.3 Differentiate $y = \frac{(x+3)^5(x^2+7x)^8}{x(x^2+5)^3}$, where $x > 0$.[3]

[3] The condition $x > 0$ assures that $(x+3)$, (x^2+7x), and x are all positive.

$$y = \frac{(x+3)^5(x^2+7x)^8}{x(x^2+5)^3}$$

$$\ln y = \ln \frac{(x+3)^5(x^2+7x)^8}{x(x^2+5)^3}$$ Take the natural logarithm of each side.

$$\ln y = \ln[(x+3)^5(x^2+7x)^8] - \ln[x(x^2+5)^3]$$ Use the fact that $\ln \frac{a}{b} = \ln a - \ln b$.

$$\ln y = 5\ln(x+3) + 8\ln(x^2+7x) - \ln x - 3\ln(x^2+5)$$ Use $\ln(ab) = \ln a + \ln b$ and $\ln a^b = b \ln a$.

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}[5\ln(x+3) + 8\ln(x^2+7x) - \ln x - 3\ln(x^2+5)]$$ Differentiate each side.

$$\frac{1}{y}\frac{dy}{dx} = \frac{5}{x+3} + \frac{8(2x+7)}{x^2+7x} - \frac{1}{x} - \frac{3(2x)}{x^2+5}$$ Apply the Chain Rule.

$$\frac{dy}{dx} = \left[\frac{5}{x+3} + \frac{16x+56}{x^2+7x} - \frac{1}{x} - \frac{6x}{x^2+5}\right] y$$ Solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \left[\frac{5}{x+3} + \frac{16x+56}{x^2+7x} - \frac{1}{x} - \frac{6x}{x^2+5}\right] \frac{(x+3)^5(x^2+7x)^8}{x(x^2+5)^3}$$ Replace y by its expression in x.

Ugly, but not as painful as differentiating this using the Quotient and Product Rules.[4] ◆

PROBLEMS FOR SECTION 17.2

1. Find $f'(x)$.
   (a) $f(x) = 2x^x$, where $x > 0$
   (b) $f(x) = 5(x^2+1)^x$
   (c) $f(x) = (2x^4+5)^{3x+1}$

2. Find $f'(x)$.
   (a) $f(x) = 3 \cdot 2^x + 2 \cdot x^3 + 3 \cdot x^{2x+3}$, where $x > 0$
   (b) $f(x) = x(2x^3+1)^x + 5$, where $x > 0$

3. Find $g'(t)$.
   (a) $g(t) = \frac{2^t}{t^{2t}}$, where $t > 0$
   (b) $g(t) = \ln(t+1)^{t^2+1}$, where $t > -1$

4. Find $\frac{dy}{dx}$ using logarithmic differentiation. You need **not** simplify.
   (a) $y = x^{\ln \sqrt{x}}$, where $x > 0$
   (b) $y = \frac{xe^{5x}}{(x+1)^2\sqrt{x-2}}$, where $x > 0$
   (c) $y = (e^{2x})(x^2+3)^5(2x^2+1)^3$

5. Suppose $y = f(x)g(x)$, where $f(x)$ and $g(x)$ are positive for all x. Use logarithmic differentiation to find $\frac{dy}{dx}$. Verify that your result is simply the Product Rule.

[4] If you ever forget the Product or Quotient Rules but remember the derivative of $\ln x$, you can use logarithmic differentiation to reconstruct the other rules for yourself.

6. Suppose $y = \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are positive for all x. Use logarithmic differentiation to find $\frac{dy}{dx}$. Verify that this is the same result you would get had you used the Quotient Rule.

7. If you felt so inclined, you could come up with a "rule" for taking the derivative of functions of the form $f(x)^{g(x)}$ where $f(x)$ is positive. You might call it "the Tower Rule" since you have a tower of functions, or you might think of a more descriptive name. In any case, what would this rule be?

17.3 IMPLICIT DIFFERENTIATION

When using the process of logarithmic differentiation, we differentiate an equation in which y is not explicitly expressed as a function of x. Logarithmic differentiation is a special case of the broader concept of **implicit differentiation**, a concept with far-reaching implications and applications. The basic idea is that we can find $\frac{dy}{dx}$ even when y is not given explicitly as a function of x. We differentiate both sides of the equation that relates x and y, applying the Chain Rule to differentiate terms involving y because y varies with x.

Implicit differentiation is an important concept; we'll begin with a very straightforward example to illustrate what is going on.

EXAMPLE 17.4 Consider the circle of radius 2 centered at the origin.[5] It is given by $x^2 + y^2 = 4$. Find the slope of the line tangent to the circle at the following points.

(a) $(1, \sqrt{3})$
(b) $(1, -\sqrt{3})$

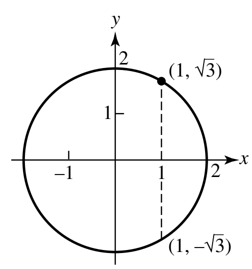

Figure 17.1

SOLUTION Although y is not a function of x, it can be expressed as two different functions of x.

$$y = \sqrt{4 - x^2} \text{ (the top semicircle)} \quad \text{and} \quad y = -\sqrt{4 - x^2} \text{ (the bottom semicircle)}$$

Each of these functions gives y explicitly as a function of x. One approach is to differentiate the former expression to get information about the point $(1, \sqrt{3})$ and the latter to deal with

[5] This circle is the set of all points a distance 2 from the origin. If (x, y) is a point on this circle, then the distance formula tells us that $\sqrt{(x-0)^2 + (y-0)^2} = 2$. And conversely, if (x, y) satisfies the equation $\sqrt{x^2 + y^2} = 2$, then (x, y) is a point on the circle. Therefore, $x^2 + y^2 = 4$ is the equation of the circle.