Systematic approach to solving games

1) Write down the game (when feasible). Fill in payoffs for each cell of the matrix or terminal node.
   a) *Normal form* if the game is simultaneous move without multiple types.
   b) *Bayesian Normal Form* if the game is simultaneous move with multiple types.
   c) *Extensive Form* if the game is sequential.

2) What is each player’s strategy set?
   a) First, determine what each player’s information sets are.
   b) A player’s strategy must specify what action to take in each of the player’s information sets.

3) When possible, eliminate *strictly* dominated strategies.\(^1\)
   a) Do not eliminate *weakly* dominated strategies.
   b) Do not eliminate strictly dominated *actions* (yet).

4) Determine what equilibrium concept to use:

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<table>
<thead>
<tr>
<th>Simultaneous move</th>
<th>Sequential move</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single type</td>
<td>Perfect info</td>
</tr>
<tr>
<td>Multiple types</td>
<td>Not perfect info</td>
</tr>
<tr>
<td>No weakly dominated strategies</td>
<td>BNE</td>
</tr>
<tr>
<td>Weakly dominated strategies</td>
<td>SPNE</td>
</tr>
</tbody>
</table>
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5) Solve. Remember that an equilibrium specifies the players’ strategies (their actions at all their information sets, not just the information sets that are reached in equilibrium). If asked for the equilibrium *outcome*, you can just specify what actions get played in equilibrium, and the resulting payoffs.

   a) *Nash equilibrium*:
      i) Find each player’s best-response function, which gives that player’s optimal strategy, given the strategies played by the other players. (In the normal form, this entails simply underlining best responses.)
      ii) Solve best-response functions simultaneously to get the set of pure-strategy Nash equilibria. (In the normal form, this is just finding cells where all payoffs are underlined.)
      iii) If there is more than one pure-strategy Nash equilibrium, look for equilibria in mixed strategies: if a player is mixing, she must be indifferent between all pure strategies that she plays with positive probability.

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\(^1\) This could theoretically cause problems if you’re solving PBE without refinements. I wouldn’t worry about it, though.
\(^2\) You may want to use PBE in cases where there are not multiple types. PBE more generally allows us to deal with beliefs.
b) **Trembling-hand perfect Nash equilibrium:**
   i) Solve for the set of Nash equilibria, as above.
   ii) Eliminate weakly dominated strategies from consideration either as pure strategies or as parts of mixed strategies (but do not iteratively eliminate weakly dominated strategies).\(^3\)

c) **Bayes-Nash equilibrium:**
   i) Solve exactly as you would solve for Nash equilibria, but expected payoffs will be a function of the probabilities over types.

d) **Backwards induction Nash equilibrium:**
   i) Find and mark best responses at the terminal decision nodes (the decision nodes closest to the terminal nodes). When there are ties, consider all tied options.
   ii) “Trim the tree”: Given the best responses at the terminal decision nodes, find best responses at the decision nodes just before the terminal decision nodes.
   iii) Iterate until you have found best responses at all nodes.

e) **Subgame perfect Nash equilibrium:**
   i) Identify all subgames. A subgame starts at a singleton node, contains all successor nodes, and does not split information sets.
   ii) Find the set of Nash equilibria in the terminal subgames (the subgames closest to the terminal nodes). When there are multiple Nash equilibria, consider all of them.
   iii) Using the results from (ii), identify the payoffs from reaching each terminal subgame.
      Using these payoffs, find the set of Nash Equilibria in the subgames one step before the terminal subgames.
   iv) Iterate using this type of backward induction by subgame until you have solved for the whole game.

f) **Perfect Bayesian Equilibrium** (consists of both strategies and beliefs):
   i) Eliminate strictly dominated actions in any information set.
   ii) Propose equilibrium strategy for one player (usually the sender, who has multiple types).
   iii) Compute other players’ beliefs based on the proposed play.
   iv) Calculate the best responses of other players, conditional on their beliefs.
   v) Check if any type of the first player has an incentive to deviate. (When necessary, find the set of off-path beliefs consistent with this play.)
   vi) Repeat until you have checked each possible pure strategy for the first player (not involving strictly dominated actions).
   vii) Apply Cho-Kreps Intuitive Criterion, if asked to do so: if a message is equilibrium dominated for a type, then assign probability zero to the message coming from that type.

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\(^3\) This isn’t 100% correct, but should suffice for our purposes.
Price fixing game

Inspired by: http://www.economist.com/finance/displaystory.cfm?story_id=10723511

This game has two firms in a market with inverse demand given by \( P(Q) = 120-Q \). Marginal cost for each firm is 0.

Competition proceeds in the following way:
In the first round, each firm decides whether to collude (C) or fight (F). If both firms choose C, then in the second round each must choose \( q_i = 30 \). If either firm chooses F, then competition is Cournot. Furthermore, if either firm plays F, neither firm has information about what second-round information set it is in, other than that at least one firm played F.

(a) What is the strategy set for each player?
(b) Write down the game in extensive form, with payoffs at terminal nodes. What basic assumption about players does this setup violate?
(c) What is the set of Nash equilibria of this game? Which of these equilibria are trembling-hand perfect?

Now change the game such that each firm’s choice of C or F is revealed after the first round.

(d) What is the new strategy set for each player?
(e) Alter your extensive form representation from part b to represent the change in the game.
(f) What is the most appropriate equilibrium concept to use for this game? What is the set of equilibria under this equilibrium concept?

Now there is a regulator of this market. The regulator observes price, but not individual firms’ quantities or whether the firms are colluding. After observing price, the regulator chooses how much to spend on an investigation of the market, \( I \in [0,1] \). If both firms are colluding, the regulator’s probability of proving collusion in court equals \( \sqrt{I} \) (assume that if the firms are not colluding, they will not be found guilty). If found guilty, both firms must pay a fine equal to \( f \). The regulator’s payoffs are as follows:
- If the firms are not colluding, the regulator gets payoff 1-\( I \).
- If the firms are colluding and found not guilty, the regulator gets payoff 0-\( I \).
- If the firms are colluding and found guilty, the regulator gets payoff 1-\( I \).

(g) What is the strategy set for the regulator?
(h) How many subgames does the entire game now have? What is the most appropriate equilibrium concept to use now?
(i) Using results from part f, what should the regulator do if it observes \( P \neq 60 \)? What should the regulator do if it observes \( P = 60 \)? What beliefs support these actions?
(j) Assuming firms are risk-neutral, what range of \( f \) can support an equilibrium (using equilibrium concept from part h) where both firms choose to collude? What range of \( f \) only supports equilibria where the firms do not collude?
(k) What would you change about this game to make it more realistic?
Price fixing game: what's going on in the game?

1) Write down the game: 

\[ \bar{\pi}_i = q_i(120 - q_i - q_j) = 30 \cdot 60 = 1800 \]

\[ (1800, 1800) \]

\[ q_j \rightarrow q_i(120 - q_i - q_j), \quad q_i(120 - q_i - q_j) \]

\[ \Rightarrow (q^*_i, q^*_j) = (40, 40) \]

\[ \Rightarrow (F, 30, 40; F, 30, 40) \text{ is NE} \]

a) 3 information sets for each player:

\[ S_i = \left\{ \{C, F\}, 30, q_i \in (0, \infty) \right\} \]

b) Violates perfect recall assumption: players may not remember if they played C or F.

c) Strictly dominated strategies?
c) (cont.) When is \((C, 30, q_i)\) a BR for both players?

We need \(1800 \geq u_c(F, 30, q_i | C, 30, q_i) \Rightarrow q_i :\)

Check player 1: \(1800 \geq q_1^*(120-q_1^*-q_2)\)

\[
\max_{q_1} \tilde{\pi}_1 = q_1(120 - q_1 - q_2) \Rightarrow q_1^* = \frac{120 - q_2}{2}
\]

\[
\Rightarrow 1800 \geq \frac{120 - q_2}{2} \left( 120 - \frac{120 - q_2}{2} - q_2 \right) \quad \text{why is this not 30?}
\]

\[
\Rightarrow 1800 \geq \left( \frac{120 - q_2}{2} \right)^2 \Rightarrow 42.43 \geq \frac{120 - q_2}{2} \Rightarrow q_2 \geq 35.15
\]

By symmetry, we have a set of NE:

\[
(C, 30, q_1 \in [35.15, \infty); C, 30, q_2 \in [35.15, \infty])
\]

THP? \((F, 30, 40)\) is weakly dominated by \((C, 30, 40)\)

For each player, so \((F, 30, 40)\) is not THP.

Much trickier, but \((C, 30, q_i \geq 60)\) is weakly dominated, so THP NE are \([C, 30, q_1 \in [35.15, 60]; C, 30, q_2 \in [35.15, 60]])

SPNE? Same as NE, since the only proper subgames start at 1.2 and 2.2.
(c) Let $S_i = \{C, F, \lambda, 30, q_{i1}, q_{i4}, q_{i5}\}$, where $q_{ij} \in [0, \infty)

f) Sequential, not perfect info, single type $\Rightarrow$ SPNE

6 subgames (starting at 1.1, 1.2, 1.3, 1.4, 1.5, 2.2)

NE in 2.2 and 1.2 are trivial.

1.3: $q_{i2} = q_{i3} = 40 \Rightarrow$ unique NE $\Rightarrow$ 1600, 1600

1.4: $q_{i4} = q_{i4} = 40 \Rightarrow$ NE $\Rightarrow$ 1600, 1600

1.5: $q_{i5} = q_{i5} = 40 \Rightarrow$ NE $\Rightarrow$ 1600, 1600

Taking these as given:

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>1800,1800</td>
<td>1600,1600</td>
</tr>
<tr>
<td>F</td>
<td>1600,1600</td>
<td>1600,1600</td>
</tr>
</tbody>
</table>

$\Rightarrow$ (c, 30, 40, 40, 40) for both players

and (c, 30, 40, 40, 40) for both players

are SPNE.

We could eliminate this by using THP.
g) For each price, the regulator specifies \( I(p) \in [0,1] \).

h) Now the regulator's info sets spread across the terminal nodes from before \( \Rightarrow \) 1 subgame (the whole game).

We need to deal with regulator's beliefs \( \Rightarrow \) PBE.

i) If \((C,C)\), then \( P = 60 \) \( \Rightarrow \) if \( P = 60 \), then not \((C,C)\) (beliefs)

\( \Rightarrow \) firms are not colluding, so regulator's payoff is \( 1 - I \) \( \Rightarrow I^* = 0 \)

If \( P = 60 \), then we expect \((C,C)\) from analysis in (f).

Regulator then solves:

\[
\max_{I \in [0,1]} \text{EU}(I) = \sqrt{I} (1-I) + (1-\sqrt{I})(-I)
\]

\[
= I^{\frac{1}{2}} - I^{\frac{3}{2}} - I + I^{\frac{3}{2}} = I^{\frac{1}{2}} - I
\]

FOC: \( \frac{d\text{EU}}{dI} = \frac{1}{2} I^{\frac{1}{2}} - 1 = 0 \) \( \Rightarrow \) \( 2I^{\frac{1}{2}} = 1 \) \( \Rightarrow \) \( I^* = \frac{1}{4} \)
j) Equilibrium outcome if both firms collude:

\[ C_1, C_2, q_1 = 30, q_2 = 30, I = \frac{1}{4} \]

\[ \Rightarrow \text{expected payoffs} \ (1800 - \frac{1}{2}f, 1800 - \frac{1}{2}f, \frac{1}{4}) \]

If firms do not collude, outcome is:

\[ F, F, q_1 = 40, q_2 = 40, I = 0 \Rightarrow \text{payoffs} \ (1600, 1600, 1) \]

If \( f \leq 400 \), firms are better off colluding
If \( f > 400 \), the only equilibrium has firms not colluding
(though one of the firms could play C)

K) 
- chance of being found guilty even if not guilty
- opportunity to rat out your competitor if you're colluding
- possibly change regulator's payoffs
- incorporate penalties other than fines (ie. jailing executives)
- etc.