Kelso-Crawford (Econometrica 1982), "Job Matching, Coalition Formation, and Gross Substitutes"

• m workers, n firms, indexed $i = 1, \ldots, m$ and $j = 1, \ldots, n$

• Each firm may hire as many workers as it wishes, but each worker is allowed to work at only one firm

• Workers care about wage and employer, but are indifferent about what other workers their firms hire

Payoffs

$u^i(j; s_{ij})$ is worker i's utility from matching with firm j at salary s_{ij}. Strictly increasing, continuous in s_{ij}.

$y^j(C^j)$ is firm j's gross product, where C^j is the set of indices of the workers firm j hires.

$\pi^j(C^j; s^j) \equiv y^j(C^j) - \sum_{i \in C^j} s_{ij}$ gives firm j's net profits.

The Firm's Problem

Given a set of salaries, firm j chooses what set of workers to hire.

Let $M^j(s^j)$ denote the set of solutions to

$$\max_C \pi^j(C; s^j),$$

where $s^j \equiv (s_{1j}, s_{2j}, \ldots, s_{mj})$ is the vector of salaries faced by j.

Gross Substitutes

From the standpoint of the firm, workers are gross substitutes. When the price of one worker goes up, demand for another worker should not go down...

Consider two vectors of salaries s^j and $\tilde{s}^j$ facing firm j. Let $T^j(C^j) \equiv \{i \mid i \in C^j \text{ and } \tilde{s}_{ij} = s_{ij}\}$.

Workers are *gross substitutes* if for every firm j

$$\text{if } C^j \in M^j(s^j) \text{ and } \tilde{s}^j \geq s^j, \text{ then there exists}$$

$$\tilde{C}^j \in M^j(\tilde{s}^j) \text{ such that } T^j(C^j) \subseteq \tilde{C}^j.$$

Allocations

An *allocation* is an assignment of workers to firms

$$f : \{1, \ldots, m\} \to \{1, \ldots, n\}$$

together with a salary schedule $(s_{1f(1)}, s_{2f(2)}, \ldots, s_{mf(m)})$.

An allocation is *individually rational* if no worker or firm prefers to remain unmatched (need to define unmatched payoffs).

The Core (strict)

A *strict core allocation* is an individually rational allocation

$$\left(f; s_{1f(1)}, s_{2f(2)}, \ldots, s_{mf(m)}\right)$$

such that there are no firm-set of workers combination (j, C) and salaries $r^j \equiv (r_{1j}, r_{2j}, \ldots, r_{mj})$ that satisfy

$$u^i(j; r_{ij}) \geq u^i\left[f(i); s_{if(i)}\right] \quad \text{for all } i \in C$$

and

$$\pi^j(C; r^j) \geq \pi^j(C^j; s^j)$$

with strict inequality holding for at least one member of $C \cup \{j\}$.

The Core

A *core allocation* is an individually rational allocation

$$(f; s_{1f(1)}, s_{2f(2)}, \ldots, s_{mf(m)})$$

such that there are no firm-set of workers combination (j, C) and salaries $r^j \equiv (r_{1j}, r_{2j}, \ldots, r_{mj})$ that satisfy

$$u^i(j; r_{ij}) > u^i\left[f(i); s_{if(i)}\right] \quad \text{for all } i \in C$$

and

$$\pi^j(C; r^j) > \pi^j(C^j; s^j).$$

Core vs Strict Core

The strict core is always contained within the core: if there is no coalition that improves on an allocation for at least one member, then there is no coalition that improves on the allocation for all members.

But when salaries are continuous, the two concepts are equivalent. Why?

The Salary Adjustment Process (an auction!)

R0. Firms begin by facing a set of (initially very low) salaries.

R1. Firms make offers to their most preferred set of workers. Any offer previously made by a firm to a worker that was not rejected must be honored.

R2. Workers evaluate offers and tentatively hold only their best ones and only if the offer is "acceptable."

R3. For each rejected offer a firm made, increment the feasible salary for the rejecting worker.

R4. If no new offers are made, terminate the process and implement the assignment and corresponding wages.

R5. Otherwise, return to step R1

The Salary Adjustment Process (an auction!)

R0. Firms begin by facing a set of (initially very low) salaries.

R1. Firms make offers to their most preferred set of workers. Any offer previously made by a firm to a worker that was not rejected must be honored. *Why does this commitment not cost the firm profits?*

R2. Workers evaluate offers and tentatively hold only their best ones and only if the offer is "acceptable."

R3. For each rejected offer a firm made, increment the feasible salary for the rejecting worker.

R4. If no new offers are made, terminate the process and implement the assignment and corresponding wages.

R5. Otherwise, return to step R1

Observation: "Choice Sets"

As the algorithm progresses, for firms, the set of feasible offers is reduced, as some offers are rejected.

For workers, the set of offers grows, as more offers come on the table. (Can think of workers having all offers ever made still on the table)

We have a monotonicity of offer sets, in opposite directions. Is this also true in Gale-Shapley?

Substitutes condition is crucial for this monotonicity.

This turns out to be very general, and will be the key to classifying matching and auctions in a common framework...

Termination

Claim 1: The process terminates. Why?

Final Allocation in Core

Claim 2: The final allocation is in the core. Why?

Suppose there is a coalition that improves upon the final allocation for all members.

For each worker, the firm must have never made an offer at a salary $\geq$ the salary in the coalitional improvement. (Why not?)

Hence, the firm's final salary it faced for each worker must have been $\leq$ the salary in the coalitional improvement.

What does this tell us about the firm's payoff in the final allocation?

Firm Optimality of Final Allocation

Claim 3: The final allocation is weakly preferred by every firm to any other allocation in the core.

Worker Proposing Algorithm

Alternatively, we could have modeled a worker proposing algorithm, where workers all begin by offering their services at the highest possible wage at their most preferred firm.

Gale-Shapley vs. Kelso-Crawford

Gale-Shapley	Kelso-Crawford
deferred acceptance algorithm	“auctioneer” price adjustments
stable matching	allocation in core
failure when $\exists$ complementarities	”
firm optimal element of core	”
strategic incentives	??

Roadmap

1. Lattice theory: a valuable tool for studying matchings and auctions
2. Identify common elements of two-sided matchings, place in a unified framework. (Hatfield-Milgrom)
3. Consider a model of n-sided matchings. Rather than pairs, participants are matched in "supply chains" of variable length.

Intro to Lattice Theory

Partially Ordered Sets

A partially ordered set $(X, \leq)$ is a set combined with a binary relation $\leq$ that satisfies

1. $\leq$ is transitive: $x \leq y, y \leq z \Rightarrow x \leq z$
2. $\leq$ is reflexive: $x \leq x$
3. $\leq$ is antisymmetric: $x \leq y, y \leq x \Rightarrow x = y$

Infimum and Supremum

$\bar{z} \in X$ is an *upper bound* for $Z \subset X$ if $z \leq \bar{z}$ for all $z \in Z$

$\sup(Z) \in X$ is the *supremum* (or *least upper bound*) for $Z \subset X$ if

1. $\sup(Z)$ is an upper bound for Z and
2. $\sup(Z) \leq \bar{z}$ for every upper bound $\bar{z}$ for Z.

$\underline{z} \in X$ is a *lower bound* for $Z \subset X$ if $\underline{z} \leq z$ for all $z \in Z$

$\inf(Z) \in X$ is the *infimum* (or *greatest lower bound*) for $Z \subset X$ if

1. $\inf(Z)$ is a lower bound for Z and
2. $\underline{z} \leq \inf(Z)$ for every lower bound $\underline{z}$ for Z.

Lattices

A *lattice* is a partially ordered set $(X, \leq)$ with the property that for all $x, y \in X$, the following points exist:

$$x \wedge y = \inf\{x, y\} \qquad \text{the "meet"}$$

$$x \vee y = \sup\{x, y\} \qquad \text{the "join"}$$

If the set X is finite, then $(X, \leq)$ is a *finite lattice*.

Maximum and Minimum Points of a Finite Lattice

Theorem: *Every finite lattice has a maximum point and a minimum point.*

Proof: Let $X = \{x_1, \ldots, x_n\}$. Consider elements

$$x_1 \wedge \ldots \wedge x_n \quad \text{and}$$
$$x_1 \vee \ldots \vee x_n.$$

Finite Lattice Example 1: Rectangular Grid

Finite Lattice Example 2: Power Sets

Let $X = \mathcal{P}(S)$ be the collection of all subsets of some finite set S. The order $\leq$ is set inclusion. Then

$$x \wedge y = x \cap y \quad \text{and} \quad x \vee y = x \cup y.$$

If $S = \{a, b, c\}$, then we have

Finite Lattice Example 3: Cross Product of Power Sets

Let S be a finite set, and let $X = \mathcal{P}(S)$. Define partial order $\leq$ on $X \times X$ by

$$(X_1, X_2) \leq (X_1', X_2') \quad \Leftrightarrow \quad X_1 \supset X_1' \text{ and } X_2 \subset X_2'.$$

Then $(X \times X, \leq)$ is a lattice.

Isotone Functions

Function $f : X \to Y$ is *isotone* if $\forall\ x \geq x'$, we have $f(x) \geq f(x')$.

- Intuitively, isotone means "weakly increasing."
- Note that the definition requires an order on both X and Y.
- We will mainly be interested in isotone functions $f : X \to X$ from a lattice to itself.

Tarski's Fixed Point Theorem

Theorem: Let $(X, \leq)$ be a finite lattice, and let $f : X \to X$ be isotone. Then the set of all fixed points of f is a non-empty lattice with respect to $\leq$.

Furthermore, the maximum and minimum fixed points are given by

$$\sup\{x|f(x) \geq x\} \quad \text{and} \quad \inf\{x|f(x) \leq x\}$$

respectively.

Iteratively Applying Isotone Functions

Theorem: Let $(X, \leq)$ be a finite lattice with maximum element $\bar{x}$ and minimum element $\underline{x}$. Let $f : X \to X$ be an isotone function. Then

1. For all k, $f^k(\bar{x}) \leq f^{k-1}(\bar{x})$. The sequence has a limit, which is the maximum fixed point of $f : X \to X$.

2. For all k, $f^k(\underline{x}) \geq f^{k-1}(\underline{x})$. The sequence has a limit, which is the minimum fixed point of $f : X \to X$.

Generalized Matching: Matching with Contracts

Cumulative Offers

In both the firm proposing deferred acceptance algorithm and in the Kelso Crawford algorithm, we could have let the workers choose from the *cumulative* set of offers made to them, not just from the set $\{\text{new offers}\} \cup \{\text{held offers}\}$.

• Since workers can only match to one firm, they always pick the best offer. Clearly they will never return to rejected offers.

The same holds for the worker proposing versions, but here we need responsive preferences (for Gale Shapley) and gross substitutes for Kelso Crawford.

• Upon receipt of a new worker offer, would a firm ever want to accept a worker offer it had previously rejected?

Cumulative Offer Algorithm: Pseudo Code

Call the offerers group A, and the receivers group B. Initially, members of A envision a finite set of packages X_A they can offer to members of B. Let X_B indicate the cumulative set of packages group B has been offered.

Cumulative Offer Algorithm: Pseudo Code cont.

0. Initialize $X_A = X_A, X_B = 0$
1. A make most favorable offers from X_A to members of B.
2. B consider all available offers, hold best, reject others.
3. Update X_A by removing offers that have been rejected. Update X_B by including newly made offers.
4. If no change to X_A, X_B, terminate.
5. Otherwise, return to step 1.

Feasibility and Stability

- Is the resulting outcome feasible?

- If it is feasible, it must be stable!

Stability Logic

Any offer made by a member of A strictly preferred by A must have been previously made. But any set of such offers corresponding to $b \in B$ has been rejected by b in favor of the final allocation.

Hence, final allocation is stable!

How does this logic play out in Gale-Shapley?

Monotonicity

Observe the monotonicity of X_A and X_B. Looks like the partial order of example three.

Matching with Contracts (Hatfield, Milgrom 2005)

$D =$ set of doctors

$H =$ set of hospitals

$X =$ finite set of *contracts*

Each contract (except for the null contract) is associated with exactly one doctor and exactly one hospital.

Examples:

1. $X = D \times H$
2. $X = D \times H \times Wages$
3. $X = D \times H \times Wages \times Terms$

Preferences

Doctors

1. Can sign only one contract.
2. Have strict preferences over the contracts that include them.

Hospitals

1. Can sign contracts with multiple doctors, but only one contract per doctor.
2. Have strict preferences over sets of contracts that name it, and satisfy (1).

Choice Notation

Let $X' \subseteq X$ be an arbitrary set of contracts.

$C_D(X'), C_H(X')$ are the contracts "chosen" from X'.

$R_D(X') \equiv X' - C_D(X')$ are the contracts "rejected" by doctors.

$R_H(X') \equiv X' - C_H(X')$ are the contracts "rejected" by hospitals.

Stable Allocations

An *allocation* is a set of contracts such that no doctor-hospital pair appears in more than one contract.

An allocation is stable if when these contracts are offered

1. None are unilaterally rejected
2. No single hospital can offer a different set of contracts that it prefers and all of its corresponding doctors weakly prefer.

Definition: Allocation X' is *stable* if

1. $X' = C_D(X') = C_H(X')$
2. $\nexists$ hospital h, set of contracts $X'' \neq C_h(X')$ such that

$$X'' = C_h(X' \cup X'') \subset C_D(X' \cup X'')$$

Substitutes: A Restriction on Hospital Preferences

Contracts are *substitutes* for a hospital h if

$$X' \subset X'' \Rightarrow R_h(X') \subset R_h(X'').$$

"If a contract is rejected, upon expanding the choice set it will continue to be rejected."

Equivalence to K-C Gross Substitutes Condition

$X = D \times H \times W$

In K-C, a hospital's choice set depends on the wage vector $\mathbf{w}$.

Claim: K-C gross substitutes $\Leftrightarrow$ contracts are substitutes for H.

" $\Rightarrow$ " Suppose K-C GS holds. Then expansion of the choice set

$\Leftrightarrow$ lowering of wages for some doctors D'

$\Rightarrow$ demand for doctors $D - D'$ cannot go up

$\Rightarrow$ contracts with $D - D'$ previously rejected are again rejected.

Also, contracts with D' previously rejected are again rejected.

Equivalence to K-C Gross Substitutes Condition, cont.

Claim: K-C gross substitutes $\Leftrightarrow$ contracts are substitutes for H.

"$\Leftarrow$" Suppose contracts are substitutes $\forall h \in H$. Lowering of wages for D'

$\Leftrightarrow$ expansion of choice set

$\Rightarrow$ previously rejected contracts continue to be rejected

$\Rightarrow$ h will not increase demand for any doctor in $D - D'$

Responsive Preferences

In the model where $X = D \times H$, we have

Claim: Responsive preferences $\Rightarrow$ contracts are substitutes.

Proof: exercise.

Opportunity Sets

The analysis relies on keeping track of the opportunity sets available to the parties.

X_D = set of contracts doctors believe might be available to them.

X_H = set of contracts hospitals believe might be available to them.

Generalized Doctor Offering Algorithm

Step 0: $X_D(0) = X, X_H(0) = \emptyset$.

Step t: $X_D(t) = X - R_H(X_H(t-1))$

$$X_H(t) = X - R_D(X_D(t))$$

If contracts are substitutes for hospitals, then we have the following interpretation:

$X_D(t) =$ offers not yet rejected

$X_H(t) =$ offers made to date

$X_D(t+1) \cap X_H(t) =$ offers held

Analysis of one-step iteration

Let's check an iteration:

$X_D(t) = X - R_H(X_H(t-1))$

What contracts have not yet been rejected by hospitals at time t?

- contracts never offered
- contracts that have been offered, but not rejected

That is, everything except contracts that have been offered and rejected.

Analysis of one-step iteration, cont.

For hospitals...

$X_H(t) = X - R_D(X_D(t))$

What contracts have been (cumulatively) offered to hospitals at time t?

- contracts that were just offered in t by doctors
- contracts previously rejected $(X - X_D(t))$

That is, everything except contracts available to D in this period that were not offered.

Offers Held

$X_D(t+1) \cap X_H(t)$

Why can we interpret this as offers held? These are offers that have been made to hospitals at time t, that are not yet rejected (ie still available to doctors) in time $t+1$. Hence, held.

Monotonicity of Opportunity Sets

X_H (offers made) growing...

X_D (offers not yet rejected) shrinking.

This is where substitutes comes in. That is, provided R is isotonic, we have monotonicity in t.

$X_D(t) = X - R_H(X_H(t-1))$

$X_H(t) = X - R_D(X_D(t))$

X_H growing $\Rightarrow$ $R_H(X_H)$ growing $\Rightarrow$ X_D shrinking.

Partial Order on $X \times X$

Define partial order $\leq$ on $X \times X$ by

$$(X', Y') \leq (X'', Y'') \quad \text{if} \quad X' \subset X'' \text{ and } Y' \supset Y''.$$

"First set grows, second set shrinks."

Welfare

With respect to this order

1. Hospital welfare is increasing, as hospitals choose from a larger set.
2. Doctor welfare is decreasing, as doctors choose from a smaller set.

Defining an Iteration as an isotone operator

Define operator $F : X \times X \to X \times X$ as follows:

$$\begin{aligned} F_D(X_D, X_H) &= X - R_H(X_H) \\ F_H(X_D, X_H) &= X - R_D(F_D(X_D, X_H)) \\ F(X_D, X_H) &\equiv (F_D(X_D, X_H), F_H(X_D, X_H)) \end{aligned}$$

Claim: If contracts are substitutes, then F is isotone.

Fixed Points of F

A fixed point (X_D, X_H) of F satisfies

$$X_D = X - R_H(X_H)$$

$$X_H = X - R_D(X_D).$$

But fixed points correspond exactly to stable allocations!

Stability of Fixed Points

Claim: If (X_D, X_H) satisfy

$$\begin{aligned} X_D &= X - R_H(X_H) \\ X_H &= X - R_D(X_D), \end{aligned}$$

then $X_D \cap X_H$ is a stable allocation.

Lemma: If (X_D, X_H) satisfies the system above, then

$$X_D \cap X_H = C_D(X_D) = C_H(X_H).$$

Proof:

$$X_D \cap X_H = X_D \cap (X - R_D(X_D)) = X_D - R_D(X_D) = C_D(X_D).$$

$$X_D \cap X_H = (X - R_H(X_H)) \cap X_H = R_H(X_H) - X_H = C_H(X_H).$$

Stability of Fixed Points, cont.

Suppose that $X' = X_D \cap X_H$ is not a stable allocation. Then there exists an allocation X'' that its doctors D' weakly prefer, and hospital h strictly prefers.

Each doctor $d \in D'$ weakly prefers $C_d(X'')$ to $C_d(X')$. Hence, we must have $C_d(X'') \notin R_D(X_D)$.

(Any contract $C_d(X'')$ preferred by d to $C_d(X')$ cannot lie in $R_D(X_D)$; this would mean that it was rejected in favor of $C_d(X_D) = C_d(X')$.)

Hence $C_d(X'') \in X - R_D(X_D) = X_H$, and thus $X'' \in X_H$.

"Any set of contracts weakly preferred by doctors must be available to hospitals..." (must have already been made)

Stability of Fixed Points, cont.

But then $X^H \supset X'' \Rightarrow$

$C_h(X_H) \succeq_h C_h(X'') \Rightarrow$ (from lemma)

$C_h(X') \succeq_h C_h(X'') \Rightarrow$

$C_h(X') \succ_h C_h(X'')$.

"...but this means that the hospital would have rejected it."

Fixed Points and Stability

It can further be shown that for any stable allocation X', there exists some solution (X_D, X_H) of the system of equations with $X_D \cap X_H = X'$.

Putting it all Together

Now... recall:

1. Start with maximum element $(X, \emptyset)$
2. Iteratively apply isotone operator F
3. Have convergence to a fixed point - the maximum fixed point. (recall from Tarski: fixed points form a lattice)
4. Since fixed points correspond exactly to stable matchings, and since doctor(hospital) welfare is increasing(decreasing) in this order this procedure yields the doctor optimal/hospital pessimal stable matching.

Hospital offering algorithm.

An analogous hospital offering algorithm begins from minimal element $(\emptyset, X)$ and reaches the hospital optimal/doctor pessimal stable matching.

Additional Assumption: Law of Aggregate Demand

Hospital preferences satisfy the law of aggregate demand if

$$X' \subset X'' \Rightarrow |C_H(X')| \leq |C_H(X'')|$$

Remarks:

1. Increasing the set of options increases the number of contracts selected.
2. In the Kelso Crawford model, this means that reducing wages increases the number of workers hired.

Law of Aggregate Demand in Previous Models

Theorem: In the Kelso-Crawford Model, if firm preferences satisfy the gross substitutes condition, then its choices satisfy the law of aggregate demand.

Theorem: In the college admissions model, responsive preferences satisfy the law of aggregate demand.

Proof: exercise.

Strategic Incentives: Dominant Strategies

Theorem: Suppose that hospital preferences satisfy substitutes and the law of aggregate demand. Then it is a dominant strategy for doctors to report their preferences truthfully in the doctor-offering algorithm.

When Contracts are Not Substitutes for Hospitals

Theorem: Suppose H contains at least two hospitals, which we denote by h and h'. Further suppose that R_h is not isotone, that is, contracts are not substitutes for h. Then there exist preference orderings for the doctors in D, a preference ordering for a hospital h' with a single job opening such that, regardless of the preferences of the other hospitals, no stable set of contracts exists.

Counter Example: Kojima 2007

Kojima has a counterexample to the previous theorem for general contract space X.

However, he shows that the theorem holds when the contract space is restricted to $D \times H \times W$; that is, the Kelso-Crawford model.

Summary

The matching with contracts model generalizes the following results familiar from the standard model.

1. Existence of stable matching
2. The set of stable matchings is a lattice
3. Doctor/Hospital offering mechanisms yields doctor/hospital optimal matchings.
4. Dominant strategies for proposing side
5. Other parallels...rural hospitals, adding doctors etc.