Chapter 5

Social Choice Theory

Social choice theory extends information economics to the context of group or societal decision-making. The underlying assumption is that we do not know the preferences of individuals, but we want to use them to make decisions. The overarching question is whether it is possible to design decision-making rules or systems that will both encourage individuals to reveal their preferences and then use that information to make a reasonable decision.

The general approach to the problem is first to study decision rules in the case of complete information about preferences and to identify a set of rules that satisfies particular desirable properties. Then, we turn back to the question of incentives for individuals to reveal information about their preferences to determine when it is a Nash equilibrium to reveal information honestly in voting (as an example of a rule for how societies make decisions in practice). An important observation is that the incentives to reveal information are bound up completely with how that information will be used in the societal decision. In general, a societal decision rule that has good properties for the case of complete information will also provide incentives for individuals to reveal their preferences honestly in voting. However, a societal decision rule that does not have good properties for the case of complete information about individual preferences, will provide incentives for people to conceal their preferences, i.e. by voting "dishonestly". Thus it makes sense to start with the decision rule and work backwards to incentives.

There is another underlying assumption, which is that the attributes of available alternatives are known to everyone. It is not possible, for example, that one person is an expert whose opinion is more informed than everyone else’s. The only uncertainty is how individuals value the alternatives. Throughout this discussion, we will (mostly) sidestep the question of strength of preferences. We
work with ordinal preferences for each person, including the possibility of ties.

5.1 The Social Welfare Functional

The societal decision rule, often called a social welfare functional, creates a social ranking that aggregates individual rank orders into a single order that applies overall. We can think of the social welfare functional as a rule that tells us how all alternatives should be ranked and what policies should be enacted as a function of the preferences of all the people in society. The important element of this functional is that it must be specified for all possible combinations of preferences; in essence, this functional specifies a decision rule for all possible circumstances before anyone’s preferences are known. A successful social welfare functional should then have desirable properties that hold for hypothetical preferences other than the one’s that are actually realized in practice. For this reason, much of our analysis shall take on this hypothetical form: we start with the social ranking for one set of preferences across individuals and then consider how the social ranking should vary with changes in individual preferences.

Although we would primarily be interested in the most preferred option overall - the one that would be chosen if only one can be selected - the social welfare functional goes on to provide an ordering of all alternatives when there are more than two alternatives. We assume, as in the case of individual preferences, that the social welfare functional should produce a complete and transitive set of pairwise rankings.

**Definition 1** Let \( \alpha_1, \ldots, \alpha_I \) represent the preferences of persons 1 through I. \( \alpha_i \) is a set of complete pairwise transitive preferences over available alternatives; this can also be represented as a rank-ordering of alternatives that may include ties. Then a social welfare functional \( F(\alpha_1, \ldots, \alpha_I) \) maps a set of known preferences into a societal preference ordering represented as \( \alpha_s \).

With two alternatives, \( A \) and \( B \), \( \alpha_i \) can be represented as one of three numbers:

\[
\alpha_i = \begin{cases} 
1 & \text{if } A \succ_i B \\
0 & \text{if } A \sim_i B \\
-1 & \text{if } A \prec_i B
\end{cases}
\]

Similarly,

\[
F(\alpha_1, \ldots, \alpha_I) = \begin{cases} 
1 & \text{if } A \succ_s B \\
0 & \text{if } A \sim_s B \\
-1 & \text{if } A \prec_s B
\end{cases}
\]

for this set of known preferences.
It is important to note that $F$ is defined across all possible preferences $(\alpha_1, ..., \alpha_I)$; we study the properties of $F$ as they apply to this entire set of possibilities, and not just to the realized preferences - i.e. a specific $(\alpha_1, ..., \alpha_I)$.

5.2 The Case of Two Alternatives

So what are some of the characteristics we would like our social welfare functional to have? We start by looking at the case of two alternatives.

5.2.1 Desirable Properties of the Social Welfare Functional

- **Paretian Property (or Unanimity):** If everyone prefers $A$ to $B$ (or $B$ to $A$), then this must translate into the societal preference. $F(1, 1, ..., 1) = 1$, and $F(-1, -1, ..., -1) = -1$.

- **Symmetry Among Agents (or Anonymity):** Only the aggregate set of preferences is important, not the way that those preferences are distributed across individuals. This has the flavor of an equal weighting assumption, because it precludes anyone from having more importance than anyone else. In practice, the symmetry property means that we don’t need to know who casts a particular vote (or holds a particular preference). Rather, the social decision depends only on knowing the total numbers of each vote (or preferences). This is why the symmetry property is sometimes called the “Anonymity Property” - voting can be anonymous without changing the result.

Formally, the symmetry property requires that any reordering of preferences across people (changing which individual has each set of preferences, but not affecting the overall collection of preferences among the whole population - essentially “shuffling” the preferences) does not change the social welfare functional. One way to say this is simply that if $(\alpha_1, ..., \alpha_I)$ and $(\alpha'_1, ..., \alpha'_I)$ represent the same set of preferences but in different orders, then $F(\alpha_1, ..., \alpha_I) = F(\alpha'_1, ..., \alpha'_I)$. In more elegant mathematical form, we can define a reordering of 1 through $I$ as an “onto function” $\pi(i)$, meaning that for each $i$ there is a number $j$ such that $\pi(j) = i$. Then, $(\alpha_{\pi(1)}, ..., \alpha_{\pi(I)})$ is a reordering of $(\alpha_1, ..., \alpha_I)$. By the definition of $\pi$ as an “onto function,” each $\alpha_i$ appears exactly once in the reordered preference vector. If $\pi(1) = 5$, for example, then $\alpha_5$ appears as person 1’s preference in the reordering $(\alpha_{\pi(1)}, ..., \alpha_{\pi(I)})$. With
this notation, the symmetry property can be written as follows: For any given reordering
(“onto”) function \( \pi \), \( F(\alpha_1, \ldots, \alpha_I) = F(\alpha_{\pi(1)}, \ldots, \alpha_{\pi(I)}) \).

**Example 2** With six individuals and two alternatives, the following must be true of the social
welfare functional \( F \) for it to satisfy symmetry: \( F(1, 1, 1, -1, -1, -1) = F(-1, -1, -1, 1, 1, 1) =
F(1, -1, 1, -1, 1, -1) = \text{etc...} \)

- **Neutrality Among Alternatives**: A social welfare functional is neutral among alternatives
if it does not favor any outcome. The property specifies, in particular, that if preferences
are completely reversed - those previously favoring \( A \) over \( B \) now strictly prefer \( B \) over \( A \),
and vice versa - then the social preference ordering must also be completely reversed. Thus,
if \( A \) is strictly preferred to \( B \) in societal ordering when 60% of individuals strictly prefer
\( A \) and 40% strictly prefer \( B \), then the opposite must be true of the social ordering if 60%
strictly prefer \( B \) and 40% strictly prefer \( A \). Another possibility is that the societal ranking
is indifferent between \( A \) and \( B \) when 60% strictly prefer \( A \) and 40% strictly prefer \( B \). If
so, neutrality requires indifference as well when 60% strictly prefer \( B \) and 40% strictly prefer
\( A \). (Presumably in this case, if everyone has strict preferences, then the societal preference
would be indifferent between \( A \) and \( B \) unless more than 60% strictly prefer one of the two
alternatives, but this is not literally required by the Neutrality property.)

Since the reverse of strict preference for \( A \) over \( B \) (\( \alpha_i = 1 \)) is simply \( \alpha_i = -1 \), we can represent
the neutrality property as the equation: \( F(\alpha_1, \ldots, \alpha_I) = -F(-\alpha_1, \ldots, -\alpha_I) \).\(^1\)

- **Positive Responsiveness**: Positive responsiveness applies to two combinations of \( \alpha \) and \( \alpha' \)
where \( \alpha' \) represents an unambiguous shift in preferences towards alternative \( A \) relative to \( \alpha \).
This requires three things: 1) each person who strictly prefers \( A \) to \( B \) for preferences \( \alpha \) also
strictly prefers \( A \) to \( B \) for preferences \( \alpha' \); 2) each person who is indifferent between \( A \) and \( B \)
either strictly prefers \( A \) to \( B \) or is indifferent between them for preferences \( \alpha' \); 3) at least one
person has a different preference for preferences \( \alpha' \) than for \( \alpha \).\(^2\) That is, at least one person

\(^1\)Note that non-monotonic preferences are still possible under this property. For instance, if \( B \) is preferred
(societally) to \( A \) when when 60% strictly prefer \( A \) and 40% strictly prefer \( B \), then \( A \) is preferred (societally) to \( B \)
when the opposite is true - when 60% strictly prefer \( B \) and 40% strictly prefer \( A \). This particular social welfare
functional produces nonsensical societal preferences, but it does satisfy the neutrality property.

\(^2\)There is no need to specify anything about the change in preference for anyone who strictly prefers \( B \) to \( A \)
for preferences \( \alpha \). Any change in preference for such people from \( \alpha \) to \( \alpha' \) is necessarily a shift in preference towards
alternative \( A \).
is "more in favor" of A under preferences $\alpha'$ than under preferences $\alpha'$ and no one is "more in favor" of B under preferences $\alpha'$ than under preferences $\alpha$. (This comparison between $\alpha$ and $\alpha'$ has the flavor of a Pareto comparison.)

The property of positive responsiveness requires that when $\alpha'$ and $\alpha$ can be ranked in this way, then if the social welfare functional ranks A ahead of B or ranks A and B the same for preferences $\alpha$, then it must rank A ahead of B for preferences $\alpha'$. That is, since the change of preferences from $\alpha$ to $\alpha'$ represents a shift in favor of A, the social ranking must reflect this shift. More specifically, any shift in individual preferences in favor of A is sufficient to break a tie in social ranking ($A \sim_s B$) in favor of A. There are three possibilities for the social preferences before and after the change in one person’s preference, as shown in the table below:

<table>
<thead>
<tr>
<th>Before</th>
<th>After</th>
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<tbody>
<tr>
<td>$A \succ_s B$</td>
<td>$A \succ_s B$</td>
</tr>
<tr>
<td>$A \sim_s B$</td>
<td>$A \succ_s B$</td>
</tr>
<tr>
<td>$B \succ_s A$</td>
<td>unspecified</td>
</tr>
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</table>

If A was at least tied in the social ranking, then the change in preferences ensures that A is strictly preferred thereafter. If B was strictly preferred to A in the social ranking, then the change in one person’s preferences might change the social preference (either to indifference or to strict preference for A), but this is not necessary; the result depends on the degree of preference for B over A in the original case. No matter what, however, when B is strictly preferred to A, the social ranking cannot be made any more in favor of B. Thus, Positive Responsiveness based on a change in preferences in favor of alternative A only restricts the social welfare functional for an $\alpha$ with $F(\alpha) = 0$ or $F(\alpha) = 1$.

With this as background, we can express this property of positive responsiveness in mathematical terms. To formalize the comparison of original and new preferences, we say $(\alpha'_1, \ldots, \alpha'_I) \succeq (\alpha_1, \ldots, \alpha_I)$ if $\alpha_i \geq \alpha'_i$ for each $i$. This means that with the $\alpha$ preferences, each person prefers A at least as much as with the $\alpha'$ preferences. If $(\alpha'_1, \ldots, \alpha'_I) \geq (\alpha_1, \ldots, \alpha_I)$ and $(\alpha_1, \ldots, \alpha_I) \neq (\alpha'_1, \ldots, \alpha'_I)$, then we know that the preference for at least one person is more in favor of A with the $\alpha'$ preferences than with the $\alpha$ preferences.

**Definition 3** If $F(\alpha_1, \ldots, \alpha_I) \geq 0$, (i.e. A is at least equal to B in the social ranking for preferences $\alpha$), and $(\alpha'_1, \ldots, \alpha'_I) \geq (\alpha_1, \ldots, \alpha_I)$ and $(\alpha_1, \ldots, \alpha_I) \neq (\alpha'_1, \ldots, \alpha'_I)$, then $F(\alpha'_1, \ldots, \alpha'_I) = 1$.
if \( F \) is positively responsive.

5.2.2 May’s Theorem

The four properties in the previous section enable us to prove the major result for voting with two alternatives - May’s Theorem:

**Theorem 4** A social welfare functional \( F(\alpha_1, \ldots, \alpha_i) \) corresponds to majority voting if and only if it satisfies 1) symmetry among agents; 2) neutrality among alternatives; and 3) positive responsiveness.\(^3\)

There are two parts of this theorem that we need to verify.

**Part I:** Majority voting satisfies these properties.

Majority voting is simply a rule that counts the number of people who strictly prefer \( A \) and the number who strictly prefer \( B \). If the number in favor of one is greater, then that is strictly preferred in the social ranking. If equal numbers have strict preference for \( A \) and \( B \), then the social ranking is indifferent between them.\(^4\)

Does majority voting satisfy...

A) ...Symmetry Among Agents? Obviously it does, since the count of strict preferences for \( A \) vs. \( B \) is not affected by the identities of the people with these preferences.

B) ...Neutrality Among Alternatives? Reversing the preferences for each person changes a majority in favor of \( A \) to a majority in favor of \( B \), and vice versa. If there is a tie between \( A \) and \( B \), reversal of all preferences maintains the tie. Thus, majority voting satisfies neutrality.

C) ...Positive Responsiveness? If there is a majority in favor of \( A \), that majority is only strengthened when one person changes preferences in favor of \( A \). If there is a tie, any change of preference by one person in favor of \( A \) breaks the tie. Thus, majority voting is positively responsive.

**Part II:** The only voting rule (or social welfare functional) that satisfies these three properties is majority voting.

\(^3\)We don’t worry about the Paretian property, because it is implied by positive responsiveness combined with neutrality.

\(^4\)Anyone who is indifferent between \( A \) and \( B \) is viewed as either casting 1/2 vote for \( A \) and 1/2 vote for \( B \) or as abstaining from a vote between \( A \) and \( B \).
Consider a set of individual preferences $\alpha$ where an equal number of people have strict preference for A over B and for B over A. We will show that symmetry and neutrality require that the societal ranking must be indifferent between A and B: $F(\alpha) = 0$. Suppose that we reorder the preferences by matching each person who strictly prefers A with a person who strictly prefers B and exchanging their preferences. This reordering reverses the preferences for anyone who has a strict preference and does not change the preferences for anyone who is indifferent between A and B. That is, this reordering changes the preferences from $\alpha$ to $-\alpha$. By neutrality among alternatives, $F(-\alpha) = -F(\alpha)$. But by symmetry among agents, this reordering cannot change the societal ranking of A and B so $F(-\alpha) = F(\alpha)$. The only way that $F(-\alpha) = -F(\alpha)$ and $F(-\alpha) = F(\alpha)$ can be satisfied simultaneously is for $F(\alpha) = 0$.

This argument implies that the social ranking is indifferent between A and B whenever an equal number of people have strict preferences for each of the two alternatives. Consider a set of preferences $\alpha'$ where more people have a strict preference for A than have a strict preference for B. We can create this preference by starting from a set of preferences where an equal number of people have strict preferences for each alternative and then changing some individual preferences from indifference to strictly preferring A. By positive responsiveness, the first such change in preferences in favor of A changes the societal ranking from indifference to a strict preference for A, and then any subsequent change in preferences maintains that strict preference. Therefore, the social ranking is a strict preference for A over B for any preferences $\alpha'$ where a majority of the people with a strict preference have a strict preference for A. By construction, the social ranking corresponds to majority voting.

This proof shows that the combination of these three properties requires the social welfare function to correspond to majority voting. In fact, each one of the three properties is necessary to the conclusion.

Without symmetry, there are many rules that satisfy the other two properties. For example, we could have a lexicographic decision rule, where a strict preference by person 1 determines the social ranking. If person 1 is indifferent between A and B, then a strict preference from person 2 determines the social ranking, and so on. This rule satisfies neutrality and positive responsiveness. But it clearly is not symmetric, since we attach special power to person 1’s preference.

Without neutrality, the social welfare functional could favor one alternative over the other - e.g. strict preference for A in the social ranking unless at least two-thirds of individuals have a strict preference for B. Despite violating neutrality, this rule still satisfies symmetry and positive
Without positive responsiveness, some majorities could still produce \( A \sim B \). For example, it could be that \( A \) and \( B \) are equal in the social ranking unless there is unanimity; this is a symmetric and neutral rule, but many votes could be changed without breaking the tie, thus violating positive responsiveness.

5.2.3 Implementation: Voting Equilibrium in the Two Alternative Case

Now that we have identified majority voting as a likely voting rule with two alternatives, we turn our attention to the Nash equilibrium when there is incomplete information for individual preferences over two alternatives. The desirable properties of majority voting translate into incentives for individuals to vote honestly when there are only two alternatives. Here, we view a majority vote election as a game where individuals submit simultaneous votes (or they abstain) and society enacts the alternative with the most votes. We assume that the voters’ preferences are unknown to one another, and that those who are indifferent abstain, since they have nothing to gain or lose by voting. We wish to verify that honest voting is a Nash equilibrium of this majority vote game.

**Theorem 5** With a majority voting decision rule, it is a dominant strategy (Nash equilibrium) for voters to vote honestly.

Because of the property of positive responsiveness, each voter recognizes that a vote for his/her favorite alternative can only help alternative to be enacted. Though there are many instances in which a single vote does not matter, a voter’s best response depends only on the outcome in the cases where her vote is pivotal, meaning that it can alter the outcome of the election.

**Proof.** Suppose that voter \( i \) has a strict preference for \( A \) over \( B \). Voter \( i \)’s vote will only have an effect on the outcome (holding everyone one’s strategy as fixed, of course) if there is a difference of no more than one vote between \( A \) and \( B \) among other voters. One vote for \( A \) can elevate \( A \) from a vote behind into a tie with \( B \), or could elevate \( A \) from a tie with \( B \) into a victory. Voting for \( B \) could only hurt \( A \)’s chances of being adopted. Thus, it is a weakly dominant strategy for person \( i \) to vote for \( A \).\(^5\) □

\(^5\) We haven’t specified the societal tie-breaking rule, but a coin flip will work just fine. Since honest voting is only a weakly dominant strategy, there are also other Nash equilibria of the majority voting game where some voters play weakly dominated strategies. For example, it is a Nash equilibrium for everyone to vote for \( A \) regardless of their preferences. In this instance, no one is pivotal, so each person can follow any voting strategy even if it seems to be
Note that this proof depends heavily on the assumption that there are no more than two alternatives. The proof points out that a vote for A can only help A, while a vote for B can only hurt A. So it is clear that the dominant strategy is to vote one’s true preference. But with three alternatives, this reasoning breaks down, as we will see in the next section.

5.3 The Case of Three or More Alternatives

Consider the voting equilibrium we examined at the end of the last section, but add a third alternative C. Suppose that person i has strict preference $A \succ_i C \succ_i B$. If the decision rule requires i to vote for one alternative, it is clear that voting for A would dominate voting for B. But it is not possible to rule out the possibility that person i should vote for C. For instance, if A has very few votes, and B and C are tied except for i’s vote, then i would prefer to vote for C than A - resulting in i’s second favorite outcome, instead of least favorite. This example is akin to the tradeoff that voters face when they prefer a third-party candidate: “Is it throwing my vote away to select Ralph Nader, when the race is likely to come down to Bush vs. Gore?”

This example indicates that the addition of a third alternative add considerably complexity to social choice theory: there is no clear result like May’s Theorem to guide the choice of a social welfare functional. Nonetheless, let us consider what kinds of properties we would want for a decision rule with more than two alternatives.

5.3.1 Desirable Properties of the Social Welfare Functional

- **Transitivity:** The social welfare functional must consist of a rank-ordering of all alternatives (thus including complete pairwise comparisons) with the possibility of ties, and social preferences will satisfy transitivity. That is, if $A \succeq_s B$, and $B \succeq_s C$, then $A \succeq_s C$. Each individual’s preferences $\alpha_i$ are likewise a complete and transitive (i.e. rational) rank-ordering of all options.

- **Unanimity:** If everyone prefers A to B (i.e. $A \succ_i B$ for all i), then the societal preference must also be for A over B: $A \succeq_s B$. And vice versa.

- **Pairwise Independence (or Independence of Irrelevant Alternatives - IIA):** The social preference for $A$ vs. $B$ depends only on individual preferences for $A$ vs. $B$. If two
sets of preferences $\alpha$ and $\alpha'$ have the same pairwise individual rankings for A vs. B, then
IIA requires that the social ranking for A and B is the same for $\alpha$ and for $\alpha'$. In words, the
social ranking of A and B does not depend on whether or not alternative C is available.

Pairwise independence is an extremely strong assumption. It says, for example, that it does
not matter whether person 1 has preferences $A \succ_1 C \succ_1 B$ or $A \succ_1 B \succ_1 C$ in terms of the
social ranking of A vs. B. In some sense, this requires us to discard information. It would
seem that $A \succ B \succ C$ should indicate a weaker preference for A over B than $A \succ C \succ B$.
Yet the two orderings are taken as equally informative for A vs. B.

This restriction of pairwise independence does apply to some voting schemes in practice -
for instance, a one-person one-vote rule would certainly not allow a voter to distinguish between
$A \succ B \succ C$ and $A \succ C \succ B$. Yet other schemes, such as the Borda Count, which gives 3 points
for a 1st place vote, 2 points for a 2nd place vote, and 1 point for a 3rd place vote, do allow for
each voter to provide information about relative preferences across all alternatives. One of the key
insights of Kenneth Arrow, whose name will come up repeatedly in this section, is that pairwise
independence is a relevant property for evaluating these more complicated voting schemes.

5.3.2 Condorcet Paradox

The heart of the problem with social choice rules with three or more alternatives is summarized by
a troublesome set of preferences. This set of preferences is known as the Condorcet Paradox,
in honor of the 19th-Century French philosopher Condorcet who first highlighted them.

<table>
<thead>
<tr>
<th>Person</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
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<tbody>
<tr>
<td>Order</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
<td>A</td>
<td>B</td>
</tr>
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</table>

For example, suppose that an individual draws independent utility values for the three alternatives at random
from a Uniform distribution (0, 100) and then ranks them from highest utility to lowest utility. Then the average
utility for the highest ranked alternative is 75, the average utility for the second-highest ranked alternative is 50, and
the average utility for the lowest-ranked alternative is 25.
Here, there are three people, with Person X strictly preferring A to B and B to C; person Y preferring B to C and C to A; and so on. Though the individual rank-orders are transitive, pairwise voting across the three people produces nontransitive preferences. That is, two people prefer A to B, two people prefer B to C, and two people prefer C to A. So, conceivably, if you started with A, a pairwise vote might approve a small expenditure to change to C. Then another pairwise vote might approve another small expenditure to change to B. And a final pairwise vote might approve a third expenditure to get back to A. Thus, Howard Raiffa labels this Condorcet cycle a “money pump.” After incurring three expenditures, society is back at its starting point. Thus, pairwise voting produces a nontransitive cycle in the Condorcet case: A ∪ B, B ∪ C, C ∪ A.

This observation already suggests that it is an insoluble problem to create a universal rule for translating individual preferences into social preferences with three or more alternatives. May’s theorem indicates that majority voting is the only "reasonable" rule for the case of two alternatives, but this example shows that the natural extension of majority voting from two to three alternatives will not work in even this simple case with three people.

One natural way to try to sidestep this problem is to rank all three alternatives equally for the Condorcet case: A ∼ s B ∼ s C. However, further examination of preferences based on the Condorcet cycle indicates an essential contradiction between unanimity and pairwise independence that makes it impossible to rank all three alternatives equally in the Condorcet cycle.

<table>
<thead>
<tr>
<th>Condorcet</th>
<th>Preference 1</th>
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<tbody>
<tr>
<td>X Y Z</td>
<td>X Y Z</td>
</tr>
<tr>
<td>1 A B C</td>
<td>1 A C C</td>
</tr>
<tr>
<td>2 B C A</td>
<td>2 C B A</td>
</tr>
<tr>
<td>3 C A B</td>
<td>3 B A B</td>
</tr>
</tbody>
</table>

Preferences 1 represent an adjusted version of the Condorcet preferences where X and Y have exchanged the order of C and B in their rankings. The connection between these preferences is that they have the same pairwise preferences for A vs. B (X and Z strictly prefer A while Y strictly prefers Z) and for A vs. C (X strictly prefers A while Y and Z strictly prefer Z). Thus, by pairwise independence, the social preference for A vs. B and for A vs. C must be the same in both cases. If the social ranking is indifferent between A, B, and C for the Condorcet case, then pairwise independence applies.

If we isolate the comparison of A vs. B, the idea is that this comparison should not be influenced by the fact...
independence requires indifference between A and B and between A and C in Preference 1. Then by transitivity, pairwise independence requires social indifference between B and C in Preference 1. Yet, unanimity requires some distinction between the preferences because C is unanimously preferred to B for Preference 1.

Pairwise independence requires that social indifference between A and C would carry over to both Preference 1 and Preference 2: \( A \sim_C C \) in Preference 1 and in Preference 2.

### Preference 2

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<th>X</th>
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<th>Z</th>
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<tbody>
<tr>
<td>1</td>
<td>B</td>
<td>B</td>
<td>C</td>
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<tr>
<td>2</td>
<td>A</td>
<td>C</td>
<td>B</td>
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<tr>
<td>3</td>
<td>C</td>
<td>A</td>
<td>A</td>
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In Preference 1, C and B are switched in ranking for individuals X and Y, but A is ranked the same by each person relative to B and C.

Suppose we want \( A \sim B \sim_C C \) in Condorcet 0. Then \( A \sim_B B \) and \( A \sim_C C \) in Preference 1. But C is preferred to B for all people in Preference 1, meaning that \( C \succ_B B \) in Preference 1. This is a contradiction with \( A \sim_B B \) and \( A \sim_C C \), so it is not possible to have \( A \sim_B B \sim_C C \) in Condorcet 0 if we also require pairwise independence. Note that we implicitly assume that the social ordering is transitive, so that we rule out the possibility of \( A \sim_C C \), \( A \sim_B B \), and \( C \succ_B B \).

With the expressed preferences in the Condorcet cycle, each alternative gets one 1st place vote, one second place vote, and one 3rd place vote. It would seem that the only reasonable social ranking would then be to place all three alternatives as equal to each other. But it is not possible to sustain this ranking and also satisfy the pairwise independence condition.

### 5.3.3 Arrow Impossibility Theorem

A similar style of analysis drives the major result of Social Choice Theory, known as the **Arrow Impossibility Theorem**. The style of proof is to show that only certain sets of social orderings that alternative C looks more attractive in Preference 1 than in Condorcet. For this thought experiment, A and B are viewed as fixed, while alternative C changes in nature. Yet, if we isolate the comparison of A vs. C, pairwise independence again only makes sense if we are thinking of A and C as fixed, while alternative B changes in nature. This represents another essential contradiction in the axiom of pairwise independence.
are consistent with pairwise independence across various sets of individual preferences. We will ultimately reach a very strong conclusion about the nature of the social welfare functional.

**Theorem 6** With three or more alternatives, the only social welfare functional that satisfies unanimity, transitivity, and pairwise independence is a dictatorship.

**Definition 7** A dictatorial social welfare functional always produces a social choice matching the individual preferences of a particular person (the dictator), regardless of the preferences of others.

We will prove a special case of the Arrow Theorem for three people and three alternatives. First, we show that if $A$ is at least tied for 1st in the social ordering for the Condorcet preferences, then $X$’s preference ordering is the social preference ordering for these preferences. Then we extend the result to show that $X$’s social preference ordering is the social ordering for all sets of possible preferences.

- **Step 1:** Suppose $A$ is at least tied for 1st in Condorcet 0. Then prove that $A \succ_s B \succ_s C$ for Condorcet 0.

By pairwise independence, we know that preferences for $A$ vs. $B$ and $A$ vs. $C$ are the same in Condorcet 0 and Preference 1. By unanimity, $C \succ_s B$ in Preference 1. Since $A$ is at least tied for 1st in Preference 1, it must be at least tied for 1st in Preference 1.

Now compare Condorcet 0 and Preference 2:

<table>
<thead>
<tr>
<th>Condorcet 0</th>
<th>Preference 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>$X$</td>
</tr>
<tr>
<td>$Y$</td>
<td>$Y$</td>
</tr>
<tr>
<td>$Z$</td>
<td>$Z$</td>
</tr>
<tr>
<td>1</td>
<td>A</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
</tr>
</tbody>
</table>

The individual preference orders for $B$ vs. $C$ and $A$ vs. $C$ are the same in both sets of preferences, so the social ordering of $B$ vs. $C$ and $A$ vs. $C$ must be the same in Condorcet 0 and Preference 2.

But $B \succ_s A$ in Preference 2 by unanimity, and $A$ is at least tied for first in Preference 1, so $A$ is at least weakly preferred to $C$ in Condorcet 0. By pairwise independence, $A$ is at least weakly
preferred to $C$ in Preference 2. Now by transitivity, since $B \succ A$ in Preference 2, $B \succ C$ in Preference 2. By pairwise independence again, $B \succ C$ in Condorcet 0. By transitivity again, since $A \succ B$ in Condorcet 0, $A \succ C$ in Condorcet 0.

*Conclusion (whew!):* $A \succ B \succ C$ for Condorcet 0.

- **Step 2:** Given Step 1, prove that the social ordering is always $A \succ B \succ C$ whenever $A \succ B \succ C$.

In an abbreviated proof, we’ll show that person X’s pairwise preferences must win out even when the other two have the opposite preferences for any pair of alternatives.

- **Case 1:** $A \succ B$, but $B \succ A$ and $B \succ A$.

One critical pair of preferences is as follows:

<table>
<thead>
<tr>
<th>Condorcet 0</th>
<th>Preference 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>$Y$</td>
</tr>
<tr>
<td>1</td>
<td>A</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
</tr>
</tbody>
</table>

We know from Step 1 that $A \succ B \succ C$ in Condorcet 0. Then:

- $C \succ B$ for Preference 3 by unanimity.

- $A \succ C$ for Preference 3 by pairwise independence, since the pairwise preferences for $A$ vs $C$ are the same in both cases.

- $A \succ B$ for Preference 3 by transitivity.

**Case 2:** $B \succ C$, but $C \succ Y A$ and $C \succ X B$.

One critical pair of preferences is as follows:

<table>
<thead>
<tr>
<th>Condorcet 0</th>
<th>Preference 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>$Y$</td>
</tr>
<tr>
<td>1</td>
<td>A</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
</tr>
</tbody>
</table>

| 1 | B | C | C |
| 2 | A | B | B |
| 3 | C | A | A |
$B \succ_s A$ for Preference 4 by unanimity.

$A \succ_s C$ for Preference 4 by pairwise independence, since $A \succ_s C$ for Condorcet 0, and the pairwise preferences for $A$ vs. $C$ are the same in both cases.

$B \succ_s C$ for Preference 4 by transitivity.

Case 3: $A \succ_X C$, but $C \succ_Y A$ and $C \succ_Z A$.

This is already true in Condorcet 0, where we know that $A \succ_s C$.

• **Step 3:** Prove that person X’s pairwise preferences also match the social preference for cases where $X$ has other orderings than $A \succ_X B \succ_X C$.

This is the critical step in the theorem. It is said that Arrow did not know that this would be true when he started the proof, and that he had a moment of inspiration, exclaiming, “The dictatorship!” Steps 1 and 2 indicate that person X acts as a dictator given one particular set of preferences for person X. Step 3 shows that the dictatorship extends to other preferences for person X.

We show that if person X strictly prefers $B$ to $A$, then $B \succ_s A$. The other cases ( $C \succ_X B$, $C \succ_X A$) are very similar. The worst case for person X occurs when $A \succ_Y B$ and $A \succ_Z B$.

We show that so long as $B \succ_X A$ then $B \succ_s A$, even in this case.

$C$ is irrelevant to the comparison between $A$ and $B$, so consider the situation where person X has $B \succ_X C \succ_X A$, while $C \succ_Y A \succ_Y B$ and $C \succ_Z A \succ_Z B$.

The social rank order for $A$ vs. $B$ for this case must be the same as for any other sets of preferences with $B \succ_X A$, $A \succ_Y B$, and $A \succ_Z B$.

Here, $C \succ_s A$ by unanimity and $B \succ_s C$ since person X has already been shown to be a dictator for comparisons of $B$ vs. $C$, and we know that $B \succ_X C$. By transitivity, $B \succ_s A$, even though both individuals Y and Z prefer $A$ to $B$.

Thus, X is a dictator for comparisons of $B$ vs. $A$. Additional analysis extends this to $B$ vs. $C$ and $C$ vs. $A$. 

189
5.3.4 Applications of the Arrow Theorem: Incentives for Strategic Voting in Practice

Most real world voting systems with three or more alternatives correspond to a non-dictatorial decision rule when everyone votes honestly. Arrow’s Theorem says that these decision rules must violate the IIA condition. That is, the decision between A and B could well depend on whether C is available. If C ranked below A and B for all individuals, it is as if C were not available as an option (A ∼C B and B ∼C A by unanimity in this case). C affects the ranking of A vs. B if it makes one of them look relatively more impressive and the other look relatively less impressive. Recognizing this element of the system, any voter who believes that the critical decision (based on everyone else’s preferences) is between A and B has an incentive to rank C in a way that makes A look better than B or that makes B look better than A.

Example 8 Suppose that there are five people with preferences as shown:

<table>
<thead>
<tr>
<th></th>
<th>L1</th>
<th>L2</th>
<th>L3</th>
<th>L4</th>
<th>L5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
<td>A</td>
<td>B</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
<td>B</td>
<td>A</td>
<td>A</td>
<td>B</td>
</tr>
</tbody>
</table>

- For the above set of preferences, consider the following possible voting rules

Simple Plurality

Each person votes once and the alternative with the highest number of votes wins, even if that alternative gets less than 50% of the votes. With honest voting, A beats B 3 to 2 when C is not available, but only ties B 2 to 2 when C is available and L5 votes for C. Thus (for most tie-breaking rules), the inclusion of C reduces A’s chance of winning, and the voting rule therefore violates IIA. The violation of IIA translates directly into an incentive for strategic voting. Given everyone else’s votes, L5 would gain by ignoring C and voting instead for A. Then A wins 3 to 2, as before.

- Conclusion: With known preferences, it is not necessarily a Nash equilibrium for everyone to vote honestly under a simple plurality system.
Sequential and Plurality Runoff

The idea of various runoff rules is to find some means - any means - to get down to two alternatives. Once we have only two alternatives, we know that honest voting is a Nash equilibrium in (weakly) dominant strategies in the final runoff election, so we can be confident of choosing the preferred alternative among those two choices. The plurality runoff selects the two alternatives with the most votes in a one-person, one-vote first stage election. The sequential runoff will have N-1 rounds of voting for N alternatives, as in each round, the option receiving the fewest votes is dropped from the election, and a new runoff occurs until there are only two remaining alternatives.

Naturally, the problem will both of these approaches is with the selection of the two alternatives for the final vote. Going back to Arrow’s Theorem, we know that there cannot be an honest voting equilibrium in the elimination stages for all preferences. (The cases that produce incentives for strategic/dishonest voting may vary from voting rule to voting rule, but there will always be some preferences for each rule that create a problem.) Individuals can anticipate the later voting outcome in the final vote and manipulate the result by changing their votes early on.

Given the preferences in the example above, honest voting in a plurality or sequential runoff (they are the same, with only three options) would give 2 votes to A, 2 votes to B, and 1 vote to C. So A and B would be selected for the runoff, which A would win 3 to 2. Interestingly, while L5 has an incentive to vote dishonestly in a simply plurality vote, there is no such incentive here. If L1 through L4 vote honestly, L5 cannot affect the selection of alternatives - A and B survive to the runoff regardless of L5’s vote. But now, L3 and L4 have an incentive to change their votes in the 1st stage. As it is, they cause their most preferred outcome, B, to survive to the runoff, where it loses to their least preferred outcome, A. It is short-sighted for them to vote for B, because B will not win in the end. Instead, either L3 or L4 should change first stage action to vote for C so that it qualifies for the runoff election in place of B. Then C will defeat A in the runoff election. For L3 and L4, this is preferable to enacting A with honest voting.

Conclusion: With known preferences, it is not necessarily a Nash equilibrium for everyone to vote honestly under a runoff system.

Comments on Plurality Rules and Third-Party Candidates: Plurality rules are generally designed with an eye towards the problem of third-party candidates. In American elections, particularly the Presidential election, there are two major party candidates and occasionally a third-party candidate (e.g. John Anderson in 1980, Ross Perot in 1992, Ralph Nader in 2000).
People who prefer the third-party candidate ($C$) to the others ($A$ and $B$) must ask themselves the questions: 1) Is this candidate $C$ going to get enough votes so that my vote for $C$ won’t be wasted? 2) If $C$ doesn’t win, will I still get a say in the choice between $A$ and $B$ if I rank $C$ in 1st place?

The simple plurality rule causes problems with both of these questions. A vote for $C$ is effectively abstaining from voting on $A$ vs. $B$, if $C$ loses. For example, there was much discussion of whether Nader voters who preferred Gore to Bush should have voted for Gore. The plurality runoff rule is aimed precisely at this case. It if seems that $A$ and $B$ are close, while $C$ has an unknown amount of support, then the first round determines whether $C$ is competitive with $A$ and $B$, allowing those who prefer $C$ to voice that preference without forfeiting a later vote for $A$ vs. $B$. Plurality runoff still runs into trouble in the example shown (which is a version of the Condorcet Cycle), because the pairwise preferences are non-transitive: $A$ beats $B$ 3 to 2, $B$ beats $C$ 3 to 2, and $C$ beats $A$ 3 to 2.

**The Borda Count**

Unlike the first two rules, the Borda Count attempts to incorporate information about the entire rank order of preferences for each individual. In one common form of the Borda Count with $n$ alternatives, each person ranks those alternatives from best to worst. The best alternative for one person gets $n$ points, the second-best gets $n$-1 points, the third-best gets $n$-2 points and so on down to 1 point for that person’s least preferred alternative. The social ranking is determined by the sum of points across all people and their preferences, with the alternative with the highest score ranked first in the social ranking.

Note that the Borda Count explicitly ignores the critical Arrow IIA property because the difference in points for an person with $A > B$ depends on how many of the other alternatives fall between $A$ and $B$ in the preference ranking for that person. The advantage of the Borda Count is that with honest voting it utilizes all of the information about preferences for each individual for each pairwise ranking. Further, the Borda Count compares alternatives by their final scores, so these comparisons are necessarily transitive. (If $A$ is strictly preferred to $B$ and $B$ is strictly preferred to $C$, then $A$’s score is greater than $B$’s scores, which is greater than $C$’s score. Thus, $A$’s score must also be higher than $C$’s score and so $A$ is strictly preferred to $C$.) The obvious disadvantage of the Borda Count is that there is now much more opportunity for individuals to manipulate the social rankings by dishonest reporting of preferences.
Example 9

\[
\begin{array}{ccc}
L1 & L2 & L3 \\
1 & A & B & A \\
2 & B & A & B \\
3 & C & C & C \\
4 & D & D & D \\
\end{array}
\]

With honest voting, \( A \) receives 2 first-place votes and 1 second-place vote for a total of 11 points \((2 \times 4 + 1 \times 3 = 11)\), while \( B \) receives 1 first-place vote and 2 second-place votes for a total of 10 points \((1 \times 4 + 2 \times 3 = 10)\). If \( L1 \) and \( L3 \) are voting honestly, then \( L2 \) can cause \( B > A \) in the social ranking by putting \( B \) in first place and \( A \) in last place. Then \( B \) continues to get 10 points, but \( A \) falls to 9 points since it only receives 1 point from \( L2 \). So honest voting is not a Nash equilibrium when the three voters are known to have these preferences.

This example demonstrates the critical flaw in the Borda Count. It is easy for one person to exaggerate the difference between two alternatives in order to promote one over the other in the social ranking. A mitigating fact is that manipulating the rankings may require considerable knowledge about the votes of everyone else. But in practice, there is often enough information to do so, and there are many conspicuous examples of manipulative voting. The Borda Count is most commonly used in sports rankings, both of players and of teams, and the following three well-known examples are all in the context of sports.

**The 1999 Most Valuable Player Voting: American League Baseball** In 1999, Pedro Martinez of the Boston Red Sox and Ivan Rodriguez of the Texas Rangers were the two strongest candidates for Most Valuable Player in the American League. A total of 28 sports writers (2 for each city with a team in the American League) voted for the award, with each choosing 10 players and ranking them in order, with higher rated players receiving more points. Although Pedro Martinez was the favorite to win the award, Rodriguez won a very close election. A subsequent analysis of individual ballots found that two writers had a disproportionate effect on the results because they did not even rank Martinez in the top ten players. While they never revealed their reasoning, it was widely believed that these two writers had intentionally downgraded Martinez so that he would not win the award.

**College Basketball Rankings** Each week during the relevant seasons, the Associated Press and *USA Today* release separate rankings of college football and college basketball teams. These
rankings are based on Borda Count votes by sportswriters (Associated Press) and coaches (USA Today). It is quite common for an individual voter to rank a local team in an unnaturally high position and thereby elevate its overall position in the rankings. There is considerable discussion each year among sports enthusiasts as to whether the poll of writers or the poll of coaches produces more honest voting and accurate rankings.

The organizations that sponsor these polls are highly sensitive to criticism. In January of 2002, Temple had been ranked approximately 40th in the college basketball rankings for several weeks in the USA Today rankings even though it had a terrible record of 6 wins and 12 losses. A USA Today staff member investigated and discovered that a single coach, Rick Majerus of the University of Utah, had been ranking Temple each week, placing it as high as 9th overall. With just 31 voters, that single vote was sufficient to keep Temple in the top 50 each week. Although Majerus had no obvious interest in promoting Temple’s ranking, he was removed as a voter in the poll.

**Figure Skating in the 2002 Winter Olympics** There was great controversy in the 2002 Winter Olympics when an investigation revealed that the some judges had colluded to fix the results of the Pairs Skating competition. In the end, the Olympic committee awarded two separate Gold Medals, one to Jamie Sale and David Pelletier, the popular choice for first place, and one to Elena Berezhnaya and Anton Sikharulidze, the winners of the original tainted voting.

There was much subsequent debate about the voting rules along with proposals for reform to promote more honest voting. Coincidentally, just days later, the results of the Women’s Figure Skating competition highlighted the fact that the Borda Count scoring system used in the Olympics violates the IIA property.

The Olympic scoring system (also used in each year’s World Championships) is based on the sum of two separate Borda Count scores, one for the “Short Program” and one for the “Long Program.” The Long Program counts twice as much as the Short Program. The overall ranking is based on “lowest ordinal score,” with ties broken in favor of the placement on the Long Program.

<table>
<thead>
<tr>
<th>Place</th>
<th>Short Program</th>
<th>Long Program</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.5 points</td>
<td>1 point</td>
</tr>
<tr>
<td>2</td>
<td>1 point</td>
<td>2 points</td>
</tr>
<tr>
<td>3</td>
<td>1.5 points</td>
<td>3 points</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>24</td>
<td>12 points</td>
<td>24 points</td>
</tr>
</tbody>
</table>
In this system, lower scores are preferred. Note that this system is equivalent to a Borda Count with Short Program scores running from 12 to 0.5 and Long Program scores running from 24 to 1 point with higher scores preferred.

The same four women skaters placed in the top four positions in both programs. The Short Program was completed first, with the following results:

**Actual Short Program Results**

<table>
<thead>
<tr>
<th>Place</th>
<th>Skater</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Michelle Kwan</td>
<td>0.5 points</td>
</tr>
<tr>
<td>2</td>
<td>Irina Slutskaya</td>
<td>1.0 points</td>
</tr>
<tr>
<td>3</td>
<td>Sasha Cohen</td>
<td>1.5 points</td>
</tr>
<tr>
<td>4</td>
<td>Sarah Hughes</td>
<td>2.0 points</td>
</tr>
</tbody>
</table>

Irina Slutskaya was the last to skate in the Long Program. At that point, Sarah Hughes was in 1st place, Michelle Kwan in 2nd place, and Sasha Cohen in 3rd place for the Long Program. Under the assumption that Slutskaya would finish 4th in the Long Program, finishing behind each of the other three, then the final standings would be:

**Tentative Standings Prior to Slutskaya’s Performance in the Long Program**

<table>
<thead>
<tr>
<th>Place</th>
<th>Skater</th>
<th>Short</th>
<th>Long</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Michelle Kwan</td>
<td>0.5 points</td>
<td>2 points</td>
<td>2.5 points</td>
</tr>
<tr>
<td>2</td>
<td>Sarah Hughes</td>
<td>2.0 points</td>
<td>1 point</td>
<td>3.0 points</td>
</tr>
<tr>
<td>3</td>
<td>Sasha Cohen</td>
<td>1.5 points</td>
<td>3 points</td>
<td>4.5 points</td>
</tr>
<tr>
<td>4</td>
<td>Irina Slutskaya</td>
<td>1.0 points</td>
<td>4 points</td>
<td>5.0 points</td>
</tr>
</tbody>
</table>

Even though Sarah Hughes was in first place in the Long Program, she would not make up her 1.5 point deficit to Michelle Kwan unless she finished two places ahead of Kwan in the Long Program. As Slutskaya started her performance, ABC, which was televising the event, showed a graphic with the current standings, indicating that Kwan was ahead of Hughes. In fact, Slutskaya finished 2nd in the Long Program, behind Hughes and ahead of Kwan. This meant that Slutskaya gained 2 points from the tentative standings while Kwan and Cohen each lost 1 point. With this change in scores, Slutskaya passed both Cohen and Kwan, while Hughes passed Kwan.

**Actual Final Standings**
<table>
<thead>
<tr>
<th>Place</th>
<th>Skater</th>
<th>Short</th>
<th>Long</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Sarah Hughes</td>
<td>2.0 points</td>
<td>1 point</td>
<td>3.0 points</td>
</tr>
<tr>
<td>2</td>
<td>Irina Slutskaya</td>
<td>1.0 points</td>
<td>2 points</td>
<td>3.0 points</td>
</tr>
<tr>
<td>3</td>
<td>Michelle Kwan</td>
<td>0.5 points</td>
<td>3 points</td>
<td>3.5 points</td>
</tr>
<tr>
<td>4</td>
<td>Sasha Cohen</td>
<td>1.5 points</td>
<td>4 points</td>
<td>5.5 points</td>
</tr>
</tbody>
</table>

With these actual final standings, Sarah Hughes won the tiebreaker ahead of Slutskaya by finishing higher in the Long Program. For television viewers and fans everywhere, however, the mystery of this competition was how Kwan could have been ahead of Hughes and yet Hughes passed Kwan without either of them skating.

*Sports Illustrated* writer Brian Cazeneuve summarized his frustration with the result, “This is the magic of fractured placement, the scoring system that sounds like a knee injury gone awry. . . . because of the scoring system that the ISU [International Skating Union] wants to abolish, Hughes needed Slutskaya to skate precisely as she did - no worse, but no better - in order to win the gold medal.” What Cazeneuve and others failed to appreciate is that the Arrow Theorem indicates that no scoring system based on ordinal placements can be free of these problems - a violation of IIA, as in this example, is to be expected.

**A Note on Arrow’s Theorem:**

Although Arrow says that no rule produces honest voting in all cases, it does not say that all rules create the same number of problems in all the same kinds of situations. For instance, it may be that plurality runoff is better than simple plurality voting, either in a particular setting or even more generally.

**5.3.5 Responses to Arrow’s Theorem**

One essential message of Arrow’s Theorem is that social choice is difficult when there are three or more alternatives. Yet, the world continues to function with rules for decisions with three or more alternatives. In many cases, such as plurality vote elections, we see at least glimmerings of the problems predicted by Arrow’s Theorem, particularly in elections in countries with three or more competitive parties. In other cases, there is little indication of controversy in decisions. In this section, we discuss two different responses that underlie decision rules that are common in practice.
Single-Peaked Preferences and the Median Voter Theorem

One approach to the Arrow Theorem is to identify cases of special preferences that get away from the problems of the Condorcet cycle. Single-peaked preferences are the most common restriction on preferences that accomplish this goal. This restriction is based on the observation that the Condorcet preferences are quite unusual and that minimal restrictions on individual preferences will guarantee the existence of acceptable rules for social choice.

Suppose that alternatives for a given policy are defined as points along the same line. Then the three alternatives would have to come in some known order, such as in the example below.

**Example 10**

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
</table>

The Condorcet preferences require 1) $A \succ B \succ C$; 2) $B \succ C \succ A$; 3) $C \succ A \succ B$.

With the ordering of alternatives in policy space as shown above, preferences 1 and 2 are plausible, but preference 3 is much less so. The reason is that to prefer $C$ the most, a person would have to tend to like policy alternatives to the far right. If so, it’s not plausible to like $A$ more than $B$, since $A$ is to the far left and $B$ is in the center.

This reasoning is formalized by the idea of **single-peaked preferences** (SPP), where every individual has an ideal point, with utility falling monotonically for that individual as policy moves in either direction from the ideal point. See Figure 5.3.5 for an example.

![Example of Single Peaked Preferences](image_url)
With strict preferences, there are 6 possible rank-orderings of three alternatives:

\[
\begin{align*}
A & \succ B \succ C & B & \succ C & \succ A & C & \succ A & \succ B \\
A & \succ C & \succ B & B & \succ A & \succ C & C & \succ B & \succ A
\end{align*}
\]

(5.1) (5.2)

If \(B\) lies in the middle of \(A\) and \(C\) in policy space, then two of these six orders (\(A \succ C \succ B\), \(C \succ A \succ B\)) can be ruled out. It is not possible to rank \(B\) lower than second place under these conditions: Someone with an ideal point to the left of \(B\) prefers \(B\) to \(C\); someone with an ideal point to the right of \(B\) prefers \(B\) to \(A\). Note that regardless of which alternative falls in the middle of the three, that alternative must be ranked at least second by all individuals, and this rules out two of the six rank orders. The problem with the Condorcet paradox is that it leads to pairwise preferences that are not transitive. The restriction to SPP eliminates this problem, a result summarized by the Median Voter Theorem.

**Definition 11** A Condorcet winner is an alternative that beats all other alternatives in pairwise voting - it wins any vote against a single other alternative.

**Theorem 12** With single-peaked preferences and an odd number of voters, the median of the voters’ ideal points is a Condorcet winner.

The restriction to an odd number of voters rules out the possibility of ties in pairwise voting.

**Example 13**

\[
\begin{array}{cccccccc}
x_1 & x_2 & x_3 & x_4 & [x_5] & x_6 & x_7 & x_8 & x_9 \\
\end{array}
\]

Suppose that the ideal points are ordered as shown, with \(x_5\) as the median. Against any policy to the left of \(x_5\), individuals 5 to 9 vote for \(x_5\). Against any policy to the right of \(x_5\), individuals 1 to 5 vote for \(x_5\). Either way, \(x_5\) wins by a vote of 5-4 against any other alternative.

The general insight of the Median Voter Theorem is that the median ideal point will win in every case, either with a “Left+Center” coalition or with a “Right+Center” coalition. The theorem is often paraphrased to say that the median voter always gets what he/she wants. This still skips two important steps: First, we must have single-peaked preferences for the term “median voter” to be meaningful. Second, if we have SPP but the voting rule is not sequential pairwise voting (say, it is simple plurality instead), there is no reason to expect the median voter’s ideal to be implemented. In other words, we have to verify that the voting process will select a Condorcet winner when it exists.
Quantitative Scoring

A second approach to the Arrow Theorem is to assume that each individual can provide quantitative information about preferences among alternatives. This approach is related to the Borda Count, which requires each placement in ranking to represent an equal difference in preference (i.e. the difference between first and second place is the same as the difference between second and third place for in each person’s ranking), except that most quantitative scoring rules in practice are more flexible than the Borda Count. Examples of quantitative scoring rules are common in sports and in application processes, such as college admissions, as we discuss in the examples below.

Professional Golf and Figure Skating after the 2002 Olympics

Most professional golf tournaments are based on results from four days of play, eighteen holes per day. By the end of the tournament, each player has completed 72 holes of play with a separate result on each one. The players could be then be given 72 separate ordinal rankings based on their hole-by-hole performances. (There would be many ties in these separate ordinal rankings because the most common result for most holes is a "par score".) Professional golf avoids problems with Arrow’s Theorem by 1) giving each player a score on each hole; 2) weighting each hole equally and summing 72 holes for each player to create a total score for the tournament. The players are then ranked for the tournament by their total scores across 72 holes.

The scoring procedure provides cardinal as well as ordinal ranking information for each of the 72 holes, much like the Borda Count. Unlike the Borda Count, because the scoring procedure for golf is determined in advance of the tournament and because each player’s performance is observed and validated by officials, the scoring procedure for golf is not obviously prone to manipulation.

After the controversies of the 2002 Olympics, the international figure skating association revamped its scoring rules. The new procedures put in place for major competitions in 2004 are based on quantitative scoring. Each elements of each routine is scored according to systematic rules. Then each competitor is given a total score that is the sum of the scores for those elements and the competitors are ranked on the basis of this total score. These rules have very similar properties to scoring for golf, with the exception that the scoring for figure skating is more subjective – different judges might give different scores to different elements in a performance, where this would not be possible with the golf scoring rules.
**College Admissions**  Many large public universities in the United States have a provision in their admissions rules that guarantees admission for students who have a combination of high grade point average and standardized test scores. Sometimes, this combination of numerical qualifications is based on a sliding scale, whereby the standardized test score requirement is relaxed for students with very high grades.\(^8\) This sliding scale reflects an implicit scoring system for student applications like those for professional sports. Each student is given one score for grades and a separate score for test scores, and these two scores are combined to give an overall ranking. However, it is not necessary to identify the specific ranking for players beyond identifying the students who qualify for admission (in California, these cutoffs are designed to include approximately 13% of high school graduates), so the scoring rules underlying these cutoffs are not stated publically.

Prior to 2003, the University of Michigan used a more elaborate scoring system for undergraduate admissions that it described in detail in its published materials. In 2003, the United States Supreme Court ruled in the case Gratz vs. Bollinger that Michigan’s scoring system was unconstitutional because it gave a certain number of points to all students in certain minority groups without regard for the information in the rest of their applications.

This decision highlights an essential flaw of all scoring rules - they cannot account for all of the possible ways that individual components of an alternative go together. That is, these rules explicitly assume that every alternative is in fact *exactly the sum of its parts*. Because they are so systematized to identify a one-dimensional score for every alternative, scoring systems guarantee the "Independence of Irrelevant Alternatives" property. Since alternatives are ranked in terms of total score, the pairwise ranking of any two alternatives is unaffected by the set of other alternatives. The inflexibility of the rules serves to eliminate subjectivity and incentives to manipulate the rankings, yet this inflexibility can also be construed as a disadvantage, as it was by the Supreme Court.\(^9\)

### 5.3.6 Concluding Thoughts and Directions for Research

There has been considerable research related to Arrow’s Theorem, with most of the work taking three different approaches:

- Relax pairwise independence in some way to produce a better formal result that does not

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\(^8\) (See for example [http://www.universityofcalifornia.edu/admissions/undergrad_adm/paths_to_adm/freshman/scholarship_reqs.html](http://www.universityofcalifornia.edu/admissions/undergrad_adm/paths_to_adm/freshman/scholarship_reqs.html) for the sliding scale that determines eligibility for enrollment at one of the branches of University of California.)

\(^9\) The decision Gratz vs. Bollinger emphasized both the inflexibility of Michigan’s scoring rule and the large number of points given to all students in particular minority groups.
require a dictatorial social decision rule.

- Consider restrictions on preferences, such as single-peakedness, and study the robustness of the Median Voter Theorem and related results. A number of papers have extended the policy space to two dimensions (e.g., social and economic) and have attempted to find conditions for a two-dimensional Median Voter Theorem. To date, no one has been able to find a multidimensional equivalent of SPP.

- Focus on the agenda-setting process and take a positive approach (“What will happen?”) rather than a normative approach (“What should happen?”) to the policy process. This is the most common approach adopted by political scientists who use game theory as a technique in the study of the legislative process.