1. Limited Liability

(a) The optimal contract for high effort, in the 2x2 case, will satisfy both the ICC and PC as equalities. Let $v(-)$ be the agent’s Bernoulli utility function, H & L denote low and high effort, $x_S$ & $x_B$ denote small and big payoffs to the firm, and $g(-)$ be the disutility of effort.

PC: $v(w_S)*P(x_S|e_H) + v(w_B)*P(x_B|e_H) - g(e_H) = u_0$

ICC: $v(w_S)*P(x_S|e_H) + v(w_B)*P(x_B|e_H) - g(e_H) = v(w_S)*P(x_S|e_L) + v(w_B)*P(x_B|e_L) - g(e_L)$

Let $w_B^*$ and $w_S^*$ solve these two equations. So, if $w_0 \leq w_S^*$, then this lower bound won’t affect wages.

b) Now, we won’t be able to satisfy the constraints as above. We should now set $w_0 = w_S$. But the new $w_S$ will be greater than the old level of $w_S^*$. To maintain the ICC, we must also increase $w_B$, since $P(x_S|e_L) > P(x_S|e_H)$. But this means that the PC will no longer bind – the agent will earn more than his/her reservation utility.

ICC: $v(w_0)*P(x_S|e_H) + v(w_B)*P(x_B|e_H) - g(e_H) = v(w_0)*P(x_S|e_L) + v(w_B)*P(x_B|e_L) - g(e_L)$

This equation now has one variable – $w_B$ – that defines the optimal contract.

c) Qualitatively, the lower bound on wages increases the reward for the good outcome that is necessary to induce high effort. Since the baseline wage is now higher, we need to create an added incentive to overcome the agent’s preference for low effort. Overall, the effect is to increase the expected wage, and increase the agent’s utility.
2. (Optional) Monotone Likelihood Ratio and Contracting

(a) The principal’s optimization problem is:

\[ \begin{align*}
\min & \quad p_{1H} w_1 + p_{2H} w_2 + p_{3H} w_3 \\
\text{s.t.} & \quad p_{1H} v(w_1) + p_{2H} v(w_2) + p_{3H} v(w_3) \geq u_0 \quad \text{(IR)} \\
& \quad p_{1H} v(w_1) + p_{2H} v(w_2) + p_{3H} v(w_3) - g(e_H) \\
& \quad \geq p_{1L} v(w_1) + p_{2L} v(w_2) + p_{3L} v(w_3) - g(e_L) \quad \text{(IC)}
\end{align*} \]

The objective is to minimize expected wage cost given that the agent chooses high effort. The first constraint is the participation constraint, and the second constraint is the incentive-compatibility constraint.

(b) The monotone likelihood ratio property is that the likelihood ratios, in this case \( p_{iL} / p_{iH} \), are decreasing in \( i \). Thus, larger outcomes are stronger signals that the agent chose high effort. If MLRP is satisfied, then the optimal wage scheme will be increasing in \( i \) (i.e., output).

(c) The likelihood ratios are 1 for \( x_1 \), 5 for \( x_2 \), and 3/7 for \( x_3 \). Thus the optimal wages will satisfy \( w_3 > w_1 > w_2 \).

(d) This is a situation where increasing effort increases the likelihood of extreme outcomes \( x_3 \) and \( x_1 \). Observing a middle outcome is actually a strong signal that low effort was taken. Thus, the optimal incentive scheme rewards both a high outcome and an low outcome relative to the middle outcome.

(e) Note that observing \( x_1 \) is a sure sign that low effort was taken. In this case, a severe punishment following \( x_1 \) will induce the agent to choose high effort. If the agent chooses high effort, however, then \( x_1 \) never occurs. Thus, a wage scheme with

\[ w_2 = w_3 = v^{-1}(u_0 + g(e_H)) \]

and \( w_1 \) very small induces high effort at first-best cost.

(f) Yes. If there is limit on the size of the punishment that can be imposed on the agent, then a low payment for \( x_1 \) may not give sufficient incentive to choose high effort. In this case, it will still be necessary to impose some risk on the agent by setting \( w_3 > w_2 \). To the extent that the contract imposes risk on the agent, the agent will need to be compensated for bearing this risk, and so the expected cost of the optimal contract will be greater than \( v^{-1}(u_0 + g(e_H)) \).

3. Contracting and Risk Aversion

(a) When both workers exert high effort, there are four possible outcomes, as listed below.

<table>
<thead>
<tr>
<th>Worker 1</th>
<th>Worker 2</th>
<th>Total Output</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Successful</td>
<td>Successful</td>
<td>2</td>
<td>( p^2 )</td>
</tr>
<tr>
<td>Successful</td>
<td>Unsuccessful</td>
<td>1</td>
<td>( p(1-p) )</td>
</tr>
<tr>
<td>Unsuccessful</td>
<td>Successful</td>
<td>1</td>
<td>( p(1-p) )</td>
</tr>
</tbody>
</table>
Given these possibilities, when both workers exert high effort, then
\[ P(y = 2) = p^2; \quad P(y = 1) = 2p(1-p); \quad P(y = 0) = (1-p)^2. \]

(b) Expected utility to the workers when both exert high effort is
\[ E(u_i(e, w) | e_H, e_H) = p v(w_2) + 2p(1-p) v(w_1) + (1-p)^2 v(w_0) - e_H. \]  \hspace{1cm} (1)

If one worker exerts effort and the other does not, then \( y = 2 \) is not possible, \( y = 1 \) occurs with probability \( p \) and \( y = 0 \) occurs with probability \( 1-p \). So expected utility to a worker who exerts low effort when the other worker exerts high effort is
\[ E(u_i(e, w) | e_L, e_H) = p v(w_1) + (1-p) v(w_0) - e_H. \]  \hspace{1cm} (2)

Combining (1) and (2), we can identify the IR and IC constraints given these conditions:

- **IR**
  \[ \begin{align*}
  E(u_i(e, w) | e_H, e_H) & \geq v(0) \\
  p^2 v(w_2) + 2p(1-p) v(w_1) + (1-p)^2 v(w_0) - e_H & \geq 0
  \end{align*} \]  \hspace{1cm} (3)

- **IC**
  \[ \begin{align*}
  E(u_i(e, w) | e_L, e_H) & \geq E(u_1(e, w) | e_L, e_H) \\
  p^2 v(w_2) + 2p(1-p) v(w_1) + (1-p)^2 v(w_0) - e_H & \geq p v(w_1) + (1-p) v(w_0)
  \end{align*} \]  \hspace{1cm} (4)

Each \( w_j \) is restricted to be non-negative and \( v(0) \) is non-negative, so \( v(w_j) \) is non-negative for each \( j \). Therefore, RHS of IC is also non-negative. Comparing (3) and (4), LHS is identical while RHS is weakly larger in (4) – whenever (4) holds then (3) does as well. This means that (IC) is a stronger condition than (IR).

(b) We can combine terms to rewrite (IC) as
\[ \begin{align*}
  p^2 v(w_2) + [2p(1-p) - p] v(w_1) - e_H & \geq (1-p) - (1-p)^2 v(w_0) \\
  (p-p^2) v(w_0) & \geq (p-p^2) v(w_0)
  \end{align*} \]  \hspace{1cm} (5)

For \( 0 < p < 1 \), the coefficient \( (p-p^2) \) is strictly positive. For any combination of wages \( (w_2, w_1, w_0) \) such that (5) holds and \( w_0 > 0 \), there is another combination of wages \( (w_2-\Delta, w_1, 0) \) satisfying (5). Since the optimal contract for the firm to induce high wages minimizes expected wages subject to (5), it is clearly preferable for the firm to choose wages \( (w_2-\Delta, w_1, 0) \) rather than \( (w_2, w_1, w_0) \) if both satisfy (IC) as represented by (5).

Intuitively, high effort reduces the probability of outcome 0, so the firm eases the incentive constraint by paying as little as possible when outcome 0 occurs.

(c) From (a) and (b), we know that the optimal contract minimizes expected wages conditional on high effort subject to the (IC) constraint (5) with \( w_0 = 0 \).

This optimization problem can be written as
\[ \text{Min} \quad p^2 w_2 + 2p(1-p) w_1 \]
\[ \text{s.t.} \quad p_2 v(w_2) + (p-2p_2) v(w_1) - eH \geq 0 \]

(NOTE: This stated problem minimizes the expected wages to one worker given the constraint for inducing high effort. Minimizing the sum of expected wages for two workers produces an equivalent problem.)

If the coefficient \((p-2p_2)\) is negative, then increasing \(w_1\) makes the constraint more difficult to satisfy and requires a corresponding increase in \(w_2\). Similar to the argument in (b), if \((p-2p_2) \leq 0\), then for any contract \((w_2, w_1, 0)\) that satisfies the constraint, there is a corresponding contract \((w_2-\delta, 0, 0)\) with \(\delta \geq 0\) that satisfies the constraint and yields lower expected wages.

Solving the inequality \((p-2p_2) \leq 0\) yields \(p (1-2p) \leq 0\) or \(p \geq 1/2\). That is, \(p \geq 1/2\), then \(p - 2p_2 \leq 0\) and the optimal contract sets \(w_0 = w_1 = 0\).

(d) When \(p < 1/2\), we can use Lagrangian methods to identify the optimal choice of \(w_1\) in terms of \(w_2\) and \(v\):

\[ L(w_1, w_2) = p_2 w_2 + 2p(1-p) w_1 - \lambda [p_2 v(w_2) + (p-2p_2) v(w_1) - eH] \]

Here, the negative sign on \(\lambda\) causes the Lagrangian objective function to increase when the constraint is violated – the appropriate effect for a minimization problem.

Differentiating with respect to \(w_1\) and \(w_2\), we have the first-order conditions

\[ \frac{\partial L}{\partial w_1} = 2p(1-p) - \lambda (p-2p_2) v'(w_1) = 0; \]

OR \[ v'(w_1) = \frac{2p(1-p)}{[\lambda (p-2p_2)]}; \quad (6) \]

\[ \frac{\partial L}{\partial w_2} = p_2 - \lambda p_2 v'(w_2) = 0; \]

OR \[ v'(w_2) = \frac{1}{\lambda} \quad (7) \]

When \(v(w) = w^{1/2}\), we can simplify (6) and (7) to

\[ 0.5 w_1^{-1/2} = \frac{2p(1-p)}{[\lambda (p-2p_2)]}; \]

OR \[ 0.25 / w_1 = \frac{2p(1-p)}{[\lambda (p-2p_2)]} \]

OR \[ w_1 = 0.25 [ \lambda (p-2p_2) / 2p(1-p) ]^2 \quad (8) \]

\[ 0.5 w_2^{-1/2} = \frac{1}{\lambda} \]

OR \[ 1 / w_2 = 4 / \lambda^2 \]

OR \[ w_2 = \lambda^2 / 4. \quad (9) \]

When \(p = 1/4\), (8) simplifies further to

\[ w_1 = 0.25 [ \lambda (1/8) / (3/8) ]^2 \]

OR \[ w_1 = \lambda^2 / 36. \quad (10) \]

Comparing (9) and (10) gives \(w_2 = 9w_1\) as the simplified first-order condition for the solution to the Lagrangian.
Further, since (IC) guarantees (IR), the only reason for the firm to offer positive wages is (IC) and so we expect that (IC) will be binding. That means the optimal contract satisfies (10) and satisfies (IC) with equality. So we have two equations in two unknowns and should be able to solve them for the wages in the optimal contract.

\[
\begin{align*}
    w_2 &= 9w_1 \quad (11) \\
    p_2 v(w_2) + (p-2p_2) v(w_1) - e_H &= 0 \\
    p_2 \sqrt{w_2} + (p-2p_2) \sqrt{w_1} &= e_H \\
    \sqrt{w_2}/16 + \sqrt{w_1}/8 &= 1 \quad (12)
\end{align*}
\]

given that \( p = 1/4, e_H = 1 \) in this problem. Substituting (11) in (12) gives

\[
\begin{align*}
    3 \sqrt{w_1}/16 + \sqrt{w_1}/8 &= 1 \\
    (5/16) \sqrt{w_1} &= 1 \\
    \sqrt{w_1} &= 256/25 = 10.24 \quad (13)
\end{align*}
\]

Substituting (13) in (11) gives \( w_2 = 9w_1 = 92.16 \) \( (14) \)

The optimal contract sets \( w_1 = 10.24, w_2 = 92.16 \).

(e) If \( p = 1/2 \) and \( v(w) = w^{1/2} \) then from (c), the optimal contract sets \( w_0 = w_1 = 0 \). As in (d), we know that (IC) is binding for the optimal contract, so (5) can be written as

\[
v(w_2)/4 = e_H = 1 \quad (15)\]

Substituting \( v(w) = w^{1/2} \) identifies an optimal contract with \( w_2 = 16 \).

With this optimal contract, the expected wage per worker is \( p_2 w_2 = 4 \), whereas each worker produces expected output equal to \( K \). So the firm is profitable if \( K = 4 \).

If \( 1 < K < 4 \) the worker’s expected output is worth more than the cost of high effort, but the firm is not profitable due to the cost (through inefficient risk sharing) of providing incentives for high effort in the optimal contract.