Problem Solutions

Example 1. Birth Order  A certain family has 6 children, consisting of 3 boys and 3 girls. Assuming that all birth orders are equally likely, what is the probability that the 3 eldest children are the 3 girls?

Solution 1. The first method is by counting. There are 3! ways to order the eldest 3 girls, and similarly 3! ways to order the youngest 3 boys. As a result, there are (3!)² = 36 ways to have the 3 eldest children being the 3 girls. Of course, there are a total of 6! ways to order the 6 children overall, so:

\[ P(3 \text{ Eldest are Girls}) = \frac{36}{6!} = \frac{1}{20} \]

Solution 2. The second method is by directly comparing what the probability of selecting a girl at each step is. The probability of the eldest child being a girl is 1/2; the probability of the second eldest being a girl is 2/5, since there are 5 children remaining, of which 2 are girls. Similarly, the probability of the third eldest being a girl is 1/4, so:

\[ P(3 \text{ Eldest are Girls}) = \frac{1}{2} \cdot \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{20} \]

Example 2. Making Teams

(a) How many ways are there to split a dozen people into 3 teams, where one team has 2 people, and the other two teams have 5 people each?

Solution. We can consider selecting 5 people from the dozen, then 5 people from the remaining 7; this automatically "selects" the remaining 2 people into a team of their own. But this overcounts: the first two teams of 5 double counts the number of ways, since we are not distinguishing the "order" in which the teams are selected. Thus, the number of ways to create the teams is:

\[ \frac{\binom{12}{5} \cdot \binom{7}{5}}{2} \]

Note that because we can select the teams in any order (i.e. pick the team of 2 first, etc.), there are other representations of this solution, but they will all end up yielding the same solution.

(b) How many ways are there to split a dozen people into 3 teams, where each team has 4 people?

Solution. Similarly, we choose 4 out of 12, then 4 out of 8, and the remaining 4 are the third team. But we are overcounting even more in this case: the ordering again does not matter, so we are overcounting by a factor of 3! (consider the simple case of \{A, B, C\} being split into "teams" of 1). Thus, the number of ways is:

\[ \frac{\binom{12}{4} \cdot \binom{8}{4}}{3!} \]

Example 3. Robberies  A city with 6 districts has 6 robberies in a particular week. Assume the robberies are located randomly, with all possibilities for which robbery occurred where equally likely. What is the probability that some district had more than 1 robbery?
Solution. The easiest way to think about this problem is to consider the complement: i.e., the probability that some district had more than 1 robbery is equal to 1 minus the probability that every district had exactly 1 robbery. There are $6^6$ ways the robberies could have occurred (imagine that you’re a thief, and you ”choose” which district to rob every evening). On the other hand, there are $6!$ ways the robberies could have occurred if each district had 1 robbery. Thus, the probability is:

$$P(\text{At Least 1 District Had More Than 1 Robbery}) = 1 - P(\text{Every District Had 1 Robbery}) = 1 - \frac{6!}{6^6} \approx 0.985$$

Example 4. Big Die I roll a 120-sided die. What is the probability that my roll is divisible by 2, 3, or 5?

Solution. Let $A_i$ be the event that my roll is divisible by $i$. We want to find $P(A_2 \cup A_3 \cup A_5)$. Whenever you see such a union, the first method of attack you want to consider is probably the Principle of Inclusion-Exclusion:

$$P(A_2 \cup A_3 \cup A_5) = P(A_2) + P(A_3) + P(A_5) - P(A_2 \cap A_3) - P(A_2 \cap A_5) - P(A_3 \cap A_5) + P(A_2 \cap A_3 \cap A_5)$$

We note that:

- $P(A_2 \cap A_3) = P(A_6) = \frac{20}{120} = \frac{1}{6}$
- $P(A_2 \cap A_5) = P(A_{10}) = \frac{12}{120} = \frac{1}{10}$
- $P(A_3 \cap A_5) = P(A_{15}) = \frac{8}{120} = \frac{1}{15}$
- $P(A_2 \cap A_3 \cap A_5) = P(A_{30}) = \frac{1}{30}$

Thus, we have every individual probability, and:

$$P(A_2 \cup A_3 \cup A_5) = P(A_2) + P(A_3) + P(A_5) - P(A_2 \cap A_3) - P(A_2 \cap A_5) - P(A_3 \cap A_5) + P(A_2 \cap A_3 \cap A_5)$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{6} - \frac{1}{10} - \frac{1}{15} + \frac{1}{30}$$

$$= \frac{11}{15} \approx 73.3\%$$

Example 5. Subsets and Combinatorics.

(a) How many subsets are there of an $n$-element set?

Solution. We can think of a ”subset” as a collection of 1s and 0s of length $n$, such that if the $i$ element is 1, then the element is included in the subset, and if the $i$ element is 0, then it is not included. Thus, we have $2^n$ possible choices, which is the number of subsets.

(b) How many subsets of size $k$ are there of an $n$-element set?
Solution. The formulaic answer is \( \binom{n}{k} \) since we are picking \( k \) elements without replacement from a set of size \( n \). Let’s try to figure out why this is the case. For the first element, we have \( n \) choices; the second, we have \( n - 1 \) choices, ..., until the last element, for which we have \( n - k + 1 \) choices. Thus, we have \( n \cdot (n - 1) \cdots (n - k + 1) = \frac{n!}{(n-k)!} \). However, this overcounts because order does not matter. Thus, by counting this way, we are counting both, say, \( \{1, 2, 3\} \) and \( \{2, 3, 1\} \) as distinct sets, which is evidently not true. Thus, we have to divide by the number of ways we can overcount, which is exactly \( k! \), so our final answer is indeed \( \frac{n!}{(n-k)!k!} = \binom{n}{k} \).

(c) Simplify the following expression using a story proof:

\[
\sum_{k=0}^{n} \binom{n}{k}
\]

Solution. What do you get if you sum up the number of ways you can pick a subset of size 0, size 1, size 2, ..., up to size \( n \) from a set of size \( n \)? Well, you get the number of all the subsets you can possibly select from the set of size \( n \), which is exactly what we calculated in (a), namely \( 2^n \). Thus:

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]


(a) You want to pick a team of \( m \) members, among which there are \( k \) leaders. Use a story proof to prove the following result:

\[
\binom{r}{m} \binom{m}{k} = \binom{r}{k} \binom{r-k}{m-k}
\]

Solution. How can you pick a team of \( m \) members among which there are \( k \) leaders? Well, you could start by picking the \( m \) members from the \( r \) total population, then picking the \( k \) leaders from the \( m \) members; but this is exactly the LHS. On the other hand, you could pick the \( k \) leaders from the population of \( r \), then pick the \( m - k \) non-leader members from the remaining population \( r - k \); but this is exactly the RHS.

(b) You want to walk from the coordinate \((0,0)\) to \((n,n)\), only taking steps in a positive direction each time. Use a story based on the number of ways to complete that walk to prove this special case of the Vandermonde Identity.

\[
\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}
\]

Solution. How many steps does it take to reach \((n,n)\) from \((0,0)\)? Exactly \(2n\) steps, if you think about it; you need \( n \) steps up and \( n \) steps to the right. So a potential path is simply selecting, of the \( 2n \) total steps, which \( n \) of them you will walk up (rather than to the right), since the remaining \( n \) steps you must walk right. This is \( \binom{2n}{n} \), or the RHS.

The LHS is a bit more complicated. To get from \((0,0)\) to \((n,n)\), we must have crossed the “diagonal” at some point, i.e. the set of points \((0,n), (1,n-1), \ldots, (n,0)\). Consider the number of ways to get from \((0,0)\) to \((0,n)\). This is equivalent to selecting \(0\) out of \(n\) steps to go up, i.e. going right every time; or \( \binom{n}{0} \) ways. Now to get from \((0,0)\) to \((n,n)\), there are again only \( \binom{n}{0} \) ways; i.e. go up on every move. Thus, there are \( \binom{n}{0} \) ways to go from \((0,0)\) to \((n,n)\) traveling via \((0,n)\). We can consider a similar situation
for going through $1, n-1$; since we need to take one step up to go from $(0, 0)$ to $(1, n-1)$, there are \( \binom{n}{1} \) ways to do so; and similarly, there are \( \binom{n}{1} \) ways to go from $(1, n-1)$ to $(n, n)$ (i.e. pick the one step that you go right). Thus, there are \( \binom{n}{1}^2 \) ways to go through $(1, n-1)$. Repeating this analysis for every one of the diagonal points yields the LHS. \[\blacksquare\]


a) What is the set of possible outcomes for the sequence of genders of their kids (the sample space)?

**Solution.** Since we want to find the sequence of genders, we need to consider each ordering separately. Thus, we have: \{$(B, B), (B, G), (G, B), (G, G)$\}.

b) Let $A$ be the event that at least one of their kids is a girl. Assuming that having a boy or a girl are equally likely, what is $P(A)$?

**Solution.** Since each of the four outcomes above is equally likely, and 3 of them contain a girl, the probability is $P(A) = \frac{3}{4}$.

c) Let $B$ be the event that at least one of their kids is a boy. What is $P(A \cap B)$?

**Solution.** The event $A \cap B$ is the event that one child is a boy and the other is a girl, and there are two ways this can happen in our sample space. Thus, $P(A \cap B) = \frac{1}{2}$.

d) What is $P(B|A)$?

**Solution.** By definition, $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/2}{3/4} = \frac{2}{3}$.

e) Are $A$ and $B$ independent events? Are $A$ and $B$ disjoint events?

**Solution.** The events are not independent. If they were, we would have $P(A) = P(A|B)$, and this is not the case (noting that $P(A|B) = P(B|A)$ by symmetry or calculation).

They are also not disjoint, because $P(A \cap B) = \frac{1}{2} > 0$. Note that if they were disjoint, they would be definitely not independent, since knowing that one event occurred would mean that the other has not.

Example 8. Biased Coins Suppose that you have two coins. One of these coins is fair, meaning that it is heads with probability $1/2$, and the other is heads with probability $3/4$.

a) Let’s say that you randomly select one of these two coins and you select each coin with probability $1/2$. If you flip this randomly selected coin once, what is the probability that you flip a heads?

**Solution.** We want the probability $P(H)$, for $H$ being the event that you flip heads. What we wish we knew is whether you picked the fair or biased coin. We can therefore condition on the event that we picked a fair coin, $F$, noting that we could have flipped a heads by picking the fair coin and doing so, or by picking the biased coin and doing so. Formally, that amounts to the following:

$$P(H) = P(H|F) \cdot P(F) + P(H|F^C) \cdot P(F^C) = \frac{1}{2} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{5}{8}$$

b) You randomly select a coin and flipped it two times, obtaining heads each time. What is the probability that you randomly selected the fair coin?
**Solution.** Suppose $HH$ is the event that we obtain two heads in two coin flips, and $F$ is again the event that we picked the fair coin. What we want is $P(F|HH)$. Anytime we see an expression like this, the first instinct should be to apply Bayes’ Rule:

$$P(F|HH) = \frac{P(HH|F)P(F)}{P(HH)} = \frac{P(HH|F)P(F)}{P(HH|F)P(F) + P(HH|F^C)P(F^C)} = \frac{1/4 \cdot 1/2}{1/4 \cdot 1/2 + 9/16 \cdot 1/2} = \frac{4}{13}$$

c) Given this information, what is now the probability that if you flip the coin one more time, it will come up as heads? (Note: This is outside the scope of this week!)

**Solution.** We want to condition everything on $HH$, since we know that the event has occurred. Thus, we want $P(H|HH)$, i.e. the probability of seeing a heads given that we have already seen two heads. The two heads tells us about the probability that we have selected the fair coin. Thus, similar to part (a), we have:

$$P(H|HH) = P(H|F, HH)P(F|HH) + P(H|F^C, HH)P(F^C|HH) = \frac{1}{2} \cdot \frac{4}{13} + \frac{3}{4} \cdot \frac{9}{13} = \frac{35}{52}$$