The spectral theorem for compact self adjoint operators and its applications.

1 Lectures 1 and 2

We discussed some history. Then Toeplitz’s theorem on summability methods, and basic facts about Fourier series including

- Fejer’s theorem on Cesaro summability of Fourier series and its applications.
- Dirichlet’s theorem on (Cauchy) summability of piecewise differentiable functions.

2 Lecture 2

Basics of Hilbert space:

- Scalar and semi-scalar products.
- Examples.
- The Cauchy-Schwarz inequality.
- The triangle inequality.
- Hilbert and pre-Hilbert spaces.
- The Pythagorean theorem.
- The theorem of Apollonius. Orthogonal projection.
- The Riesz representation theorem, which is a key tool in the course.
3 Lecture 3
The spectral theorem for compact self-adjoint operators. Including Bessel’s inequality and application to Fourier’s Fourier series via integration by parts.

4 Lectures 4 and 5
Applications of the spectral theorem for compact self-adjoint operators.
- Sobolev spaces.
- Distributions and Schwartz’s theorem.
- Rellich’s lemma.
- Elliptic differential operators.
- Gårding’s inequality.
- Consequences of Gårding’s inequality.
- Weyl’s method of orthogonal projection in potential theory.
- Extension of the basic lemmas to manifolds.
- Example: Hodge theory.

Semigroups and the spectral theorem for self-adjoint operators.

5 Lecture 6 - The Fourier transform.
- Conventions, especially about $2\pi$.
- The Schwartz space $\mathcal{S}$.
- Basic facts about the Fourier transform acting on $\mathcal{S}$.
- The Fourier transform on $L_2$.
- The Shannon sampling theorem.
- The Heisenberg Uncertainty Principle.
- Tempered distributions.
• Examples of Fourier transforms of elements of $S'$.
• The Laplace transform.
• The Mellin inversion formula.
• The spectral theorem for bounded self-adjoint operators, functional calculus form via the Fourier inversion formula.
• The Mellin transform
• Dirichlet series and their special values

6 Lectures 7-9, Semi-groups.

• The power series expansion of $e^{tA}$ when $A$ is bounded.
  – The resolvent is the Laplace transform of the semi-group.
  – Getting the semigroup from the resolvent via a contour integral.
  – The two resolvent identities.
• Unbounded operators, their resolvents and their spectra.
• Sectorial operators.
  – Definition of a sectorial operator.
  – Definition of $e^{tA}$ when $A$ is sectorial.
  – The semi-group property.
  – Bounds on $e^{tA}$.
  – The derivatives of $e^{tA}$, its holomorphic character, the limit as $t \to 0$
  – The resolvent as the Laplace transform of the semigroup.
• The spectrum of a self-adjoint operator is real.
• Equibounded continuous semi-groups.
• Using the Mellin inversion formula for the Laplace transform.
• The Hille-Yosida theorem
• The spectral theorem for unbounded self-adjoint operators from the Hille-Yosida theorem.
7 Lecture 10 - The spectrum of the Hamiltonian of the square well

- The Kato-Rellich theorem.
- The spectrum as “approximate eigenvalues”.
- The discrete spectrum and the essential spectrum.
- Weyls theorem on the stability of the essential spectrum.
- The domain of the free Hamiltonian in one dimension.
- The eigenvalues for the square well Hamiltonian.

Lectures 11-15 were an excursion into measure and integration theory.

8 Lecture 11 - Lebesgue measure

- Lebesgue outer measure.
- Lebesgue inner measure.
- Lebesgue’s definition of measurability.
- Caratheodory’s definition of measurability.
- Countable additivity. $\sigma$-fields, measures, and outer measures.
- The Borel-Cantelli lemmas.
- Basics of information theory.
- The Hewitt-Savage zero one law.

9 Lecture 12 - Constructing outer measures.

- Constructing outer measures, Method I.
- Metric outer measures, Caratheodory’s theorem.
- Constructing outer measures, Method II.
- Hausdorff measure.
- Hausdorff dimension.
10 Lecture 13 - Lebesgue integration theory.

- Real valued measurable functions.
- The integral of a non-negative function.
- Fatous lemma.
- The monotone convergence theorem.
- The space $L_1(X, \mathbb{R})$.
- The dominated convergence theorem.
- Riemann integrability.
- The Beppo-Levi theorem.
- $L_1$ is complete.
- Dense subsets of $L_1(\mathbb{R}, \mathbb{R})$.
- The Riemann-Lebesgue Lemma and the Cantor-Lebesgue theorem.
- Fubinis theorem.
- The Borel transform.

11 Lecture 14 - The Daniel-Stone integration theory.

- The Daniell Integral.
- Monotone class theorems.
- Measure and Stone’s axiom.
- Hölder, Minkowski, $L_p$ and $L_q$.
- The Radon-Nikodym Theorem.
- Duality of $L_p$ and $L_q$. 
12 Lecture 15 - Riesz representation theorems and an improved spectral theorem.

• A Riesz representation theorem for measures.
• Integration on locally compact Hausdorff spaces.
• The spectral theorem - functional calculus for bounded Borel measurable functions.
• Resolutions of the identity.
• Radon Nikodym
• Fubini’s theorem.
• The Riesz representation theorem redux.
  – Propositions in topology.
  – Proof of the uniqueness and regularity of the measure.

13 Lecture 16 - Wiener measure.

• The Big Path Space.
• The heat equation.
• Paths are continuous with probability one.
• Stroock’s imbedding in $S'$.
• Stochastic processes and Gelfand’s generalized stochastic processes.
• Gaussian measures.
• Generalities about expectation and variance.
• Gaussian measures and their variances.
• The variance of a Gaussian with density.
• The variance of Brownian motion.
• The derivative of Brownian motion is white noise.

Return to semi-groups and to the spectral theorem.
14 Lecture 17 - Convergence of semi-groups.

- A theorem of Lie,
- The Trotter product formula.
- Feynman path integrals.
- Convergence of semigroups.
- Chernoffs theorem.
- Proof of the Trotter product formula from Chernoff’s theorem.
- The Feynman Kac formula

15 Lecture 18 The $L_2$ spectral representation.

- The multiplication version of the spectral theorem.
  - The cyclic case.
  - The general case.
- Fractional powers of a non-negative operator.
- The Lax-Milgram theorem,
- Rigged Hilbert spaces.
- Semi-bounded operators and the Friedrichs extension.
- The hydrogen atom and Hardy’s inequality.

16 Lecture 19 - Return to the discrete and the essential spectrum.

- The discrete and the essential spectrum. Finite rank operators.
- A normal form for norm limits of finite rank operators.
- Compact operators.
- Hilbert Schmidt integral operators.
- Hilbert Schmidt operators in general.
- Weyls theorem on the essential spectrum.
- Applications to Schrödinger operators. Friedrich’s theorem.
17 Lecture 20

- Potentials in $L_2 \oplus L_\infty$.
- Rayleigh-Ritz and its applications.
  - Variations on the variational formula.
  - The secular equation.
- Valence.
  - Two dimensional examples.
  - The Hückel theory of hydrocarbons, the adjacency matrix of a graph.

Homeworks

1. Probability from a Hilbert space viewpoint.
2. The central limit theorem.
3. Convexity, arbitrage, and probability.
5. Representation theory of compact groups. The Peter-Weyl theorem.
6. The Perron-Frobenius theorem.
7. Conditional expectation, martingales, waiting time until a pattern.
8. The Kalman filter.