Strategic Behavior in Multi-Unit Assignment Problems: Theory and Evidence from Course Allocations

Eric Budish and Estelle Cantillon

Market Design

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The Multi-Unit Assignment Problem

- Defining features of the problem
  - Scarce resources
  - Multi-unit demand
  - Transfers are not allowed

- Example Environments
  - Students & Courses
  - Students & Interview Slots
  - Workers & Tasks
    - (Local example: professors and undergraduate theses)
  - Captains choosing a Team
These defining features make our problem special

If no scarcity → problem is trivial

If unit-demand → Random Serial Dictatorship.

- Abdulkadiroglu and Sonmez (1998) and Pathak (2006) offer arguments that RSD is procedurally fair

If transfers are allowed → Vickrey-Clarke-Groves.

- We should have in mind some institutional barrier against using cash (related to "Repugnance", Roth 2007)
The Multi-Unit Assignment Problem is Unsolved

- The only known mechanism that is anonymous, strategyproof and ex-post efficient is Random Serial Dictatorship ("all at once")
  - This is a negative result: Dictatorships result in very unequal outcomes ex-post.
  - Unacceptable to school administrations
  - Problem doesn’t go away as market gets large: no known alternative mechanisms even for a continuum

- Several recent characterization papers weaken anonymity and provide conditions under which some form of Dictatorship is "The only known mechanism that is strategyproof and ex-post efficient"
  - Ehlers & Klaus (2003), Hatfield (2004), Konishi Quint and Wako (2001), Papai (2001)
  - Results are for deterministic mechanisms with finite number of players
Overview of Today’s Talk
Theory and Evidence from the HBS Mechanism

- Nature of the incentive problem in multi-unit assignment
  - "Bidding points" mechanisms
  - "Choose one at a time" mechanisms

- Properties of a specific multi-unit assignment mechanism, used to allocate elective courses at Harvard Business School

- Evidence of strategic behavior in the HBS mechanism
  - Overreporting of popular courses
  - Congestion
  - Welfare Loss

- Evidence that HBS may nevertheless be preferable to RSD in terms of fairness and ex-ante efficiency, despite the latter’s being ex-post efficient and strategyproof
Nature of the Incentive Problem

Environment

- Set of students $S = \{s_1, ..., s_n\}$, each requires a seat in $m$ courses
- Set of courses $C = \{c_1, ..., c_K\}$, with capacities $Q = (q_1, ..., q_K)$.
- Set of students' preference orderings $P = (P_s)_{s \in S}$. Preferences are common knowledge to the students.
- Students have a vNM utility function over stochastic outcomes.
- **Responsiveness.** Students' preferences over individual courses are strict, and their preferences over bundles of courses are responsive to their underlying individual course preferences (Roth, 1985).
- **Scarcity.** There exists at least one course $c_k$ such that more than $q_k$ students rank it in their top $m$ choices.
A Bidding Points mechanism $\mathcal{M}$ proceeds as follows:

1. The mechanism announces
   - restrictions on what bid vectors are permissible (e.g. a budget constraint)
   - schedule-constraint rule

2. Students report permissible bid vectors $b = (b_s)_{s \in S}$, $b_s = (b_{s1}, \ldots, b_{sK})$

3. All $S \times K$ bids are sorted in descending order, breaking ties where necessary.

4. The mechanism proceeds through the bids in order. At each particular bid $b_{sc}$, $s$ is allocated a seat in $c$ if (i) course $c$ is not at capacity and (ii) student $s$ is not schedule-constrained from obtaining $c$. 
Bidding Points Mechanisms in Practice

- Sonmez and Unver (2004) and Krishna and Unver (2006) study the UMich mechanism in depth, and provide a lot of institutional detail.

- **University of Michigan.** Each student has the same budget (e.g. 1000 points). Ties are broken randomly, at each bid amount. Standard schedule constraints (timing, number of classes).
  - Substantially similar versions at Yale, Princeton
  - At Kellogg, the budget carries over for multiple semesters

- **Columbia.** 2 trading days. The first trading day is identical to UMich. The second trading day is for "filling gaps."
  - Substantially similar at Haas
There are several other mechanisms in practice that don’t look like Bidding Points mechanisms at first, but in fact have a similar structure:

- **Wharton.** 10+ trading days, organized as a sequential Double Auction (motivated by financial market designs).
  - The first trading day is actually identical to UMich, but without any schedule constraints.
  - In subsequent trading days, students can also submit "asks" for courses they currently own. Clearing price for a course is the (mean) price at which supply = demand.
  - By the end of the last trading day students must have a schedule that satisfies standard constraints.
  - Students can always dispose of an unwanted class to the Registrar at price 0.

- **Stanford Business School.** A multi-unit Boston Mechanism.
  - Students submit ROLs, which can be interpreted as bid vectors (30, 29, ..., 1) to fit in the class.
Bidding Points Mechanisms are not Strategyproof

**Theorem**

Suppose $M$ is a Bidding Points mechanism. Then $M$ is not strategyproof, (meaning that some students will submit bid vectors that do not reflect their ordinal preferences).

- **Proof.** Several examples in Sonmez and Unver (2004).
- Strategic issue I: don’t want to underbid for a popular course
  - Squander points, just like in the Boston Mechanism
- Strategic issue II: don’t want to underbid for an unpopular course that you like a lot
  - May fill up your schedule with courses you like less but bid more for, for strategic reasons
- Wharton mechanism probably mitigates both problems to some extent.
  - Very difficult to model, and certainly quite strategic
There are $n$ students who must make $m$ choices each, so there are $\frac{(nm)!}{m!n}$ choosing orders. Call the set of choosing orders $\mathcal{L}$. $\lambda \in \mathcal{L}$ is a mapping from $\{1, \ldots, nm\} \rightarrow S$.

A choose-one-at-a-time mechanism $\mathcal{M}$ proceeds as follows:

1. A probability distribution $\Lambda$ over $\mathcal{L}$ is announced publicly.
2. Students report rank-order lists over individual courses: $\widehat{P} = (\widehat{P}_s)_{s \in S}$.
3. A choosing order $\lambda$ is chosen according to the distribution $\Lambda$.
4. The mechanism proceeds through the choosing order $\lambda$, for each turn $t = 1, 2, \ldots, nm$. Whichever student’s turn it is to choose, $\lambda(t)$, is allocated his most preferred course in $\widehat{P}_{\lambda(t)}$ which (i) he has not yet received, and (ii) is not yet at capacity.
Special cases of the Choose One at a Time class

- **HBS Mechanism.** Each student is assigned a single random priority number:
  - Rounds 1, 3, 5, ...: Students choose one at a time in *ascending* order
  - Rounds 2, 4, 6, ...: Students choose one at a time in *descending* order

- **Random Serial Dictatorship.** Each student is assigned a single random priority number. Each student gets $m$ consecutive choices when it is her turn.

- **Dictatorship.** In every ordering that occurs with positive probability, each student gets $m$ consecutive choices.

- **Deterministic.** $\Lambda$ is degenerate. (Manea (2007) considers a variant of this mechanism, in extensive form).
Remarks on the Choose One at a Time Class

- **Remark 1.** When \( m = 1 \), HBS = RSD.
  - The HBS mechanism is strategyproof for \( m = 1 \), Bidding Points mechanisms are not.

- **Remark 2.** There are two important, and distinct, differences versus the UMich Bidding Points mechanism
  - **Difference 1.** At each turn a course is allocated. Students can’t "squander" their turns by asking for a course already at capacity.
  - **Difference 2.** The HBS mechanism is less expressive. There is no way to signal the intensity of preferences.
  - Advocates of Bidding Points mechanisms tend to focus on Difference 2, and ignore Difference 1.
Theorem

Suppose $M$ is a choose-one-at-a-time mechanism. Then $M$ is strategyproof if and only if it is a Dictatorship.

Proof. If $m = 1$ the result follows immediately. Suppose $m \geq 2$.

(if) For any realization $\lambda$ each student’s $m$ choices are consecutive.

Since preferences are responsive, he can do no better than simply to select in order his $m$ most-preferred courses that are available at the (random) time of his first choice. Truthful play accomplishes exactly this.
Proof, cont. (only if) Suppose \( M \) is not a Dictatorship. There exists a \( \lambda^*, s, k \) such that in positive-probability realization \( \lambda^* \) student \( s \)’s \( k \) and \( k + 1 \) choices are non-consecutive. Assume the other students play truthfully. Construct preferences and capacities as follows:

- \( P_s : c_1, c_2, ..., c_m, c_{m+1}, \text{any} \)
- \( P_{-s} : \) such that \( c_1, c_2, ..., c_k, c_{k+2}, c_{k+3}, ..., c_{m+1} \) never reach capacity under truthful play, and \( c_{k+1} \) reaches capacity between \( s \)’s \( k^{th} \) and \( k + 1^{th} \) choices in \( \lambda^* \).

Consider the deviation \( P_s^* \) in which student \( s \) switches the order of \( c_k \) and \( c_{k+1} \). Both \( P_s \) and \( P_s^* \) obtain \( c_1, c_2, ..., c_k, c_{k+2}, c_{k+3}, ..., c_m \) with probability one.

- \( P_s^* \) yields a higher probability of obtaining \( c_{k+1} \) instead of \( c_{m+1} \), which increases his expected utility since \( P_s \) is responsive. This contradicts strategyproofness.
Interpretation of Theorem 2

- Multi-Unit Demand is what creates the possibility of "gaps between choices" during which popular courses can reach capacity, hence creating incentive to act strategically.
- This incentive persists as the market grows large.
We aim to prove the following conjecture re the HBS mechanism:

1. "students overreport 'popular' courses in equilibrium", for a reasonable definition of popularity in terms of the primitive preferences.

2. "congestion increases in equilibrium", with congestion defined in terms of the round at which courses reach capacity.

There are two difficulties with these claims:

1. Multiple equilibria. Congestion can be lower than under truthful play.
2. Overflow demand. Overflow demand creates the possibility that the round at which a course reaches capacity is stochastic.
Properties of the HBS Mechanism

Incentive to Overreport "Popular" Courses

- **Example 1.** $n$ students who require $m = 2$ courses, $K = 4$ courses with $n/2$ seats each.
  - $\frac{n}{2}$ are $P_1 : c_1, c_2, c_3, c_4$
  - $\frac{n}{2}$ are $P_2 : c_4, c_1, c_2, c_3$. Truthful play → obtain $\{c_4, c_3\}$ or $\{c_4, c_2\}$

- Truthful play is not a best-response for the $P_2$ types. The following is a Nash Equilibrium:
  - $P_1$ play $\overline{P_1} : c_1, c_2, c_3, c_4$.
  - $P_2$ play $\overline{P_2} : c_1, c_2, c_4, c_3$. Strategic play → obtain $\{c_1, c_4\}$ or $\{c_2, c_4\}$

- Remark 1: The $P_2$ students are overreporting the "popular" courses $c_1$ and $c_2$ and underreporting the "unpopular" course $c_4$

- Remark 2: Congestion increases relative to truthful play. Courses $c_1$ and $c_2$ now both reach capacity in the first round.
Properties of the HBS Mechanism

Multiple Equilibria

Example 2. \( n \) students require \( m = 2 \) courses. Courses have \( 0.4n \) seats each. Courses \( c_1, c_2, c_3 \) have excess demand, all other courses do not.

| \( \frac{3n - \varepsilon}{2} \) | \( P_1 \) | \( c_1, c_2, \text{other} \) |
| \( \frac{3n - \varepsilon}{2} \) | \( P_2 \) | \( c_2, c_1, \text{other} \) |
| \( 0.4n + \varepsilon - \delta \) | \( P_3 \) | \( c_3, \text{other} \) |
| \( \delta \) | \( P_4 \) | \( c_3, c_1, \text{other} \) |
| \( \delta \) | \( P_5 \) | \( c_3, c_2, \text{other} \) |

- For \( n \) large, \( \varepsilon, \delta \) small enough, \( \delta > \varepsilon > 0 \), \( \exists \) (at least) 2 equilibria:
- Equilibrium 1: All players are truthful. \( c_3 \) reaches capacity in the first round, \( c_1 \) and \( c_2 \) reach capacity in the second round.
- Equilibrium 2: \( P_1, P_2, P_3 \) are truthful. \( P_4 \) and \( P_5 \) switch the order of their first two choices.
  - Now, \( c_3 \) reaches capacity in the second round. Congestion is lower.
  - But, types \( P_4 \) and \( P_5 \) still get \( c_3 \) with probability that goes to 1 as \( \varepsilon \rightarrow 0^+ \).
Properties of the HBS Mechanism

Truthful Play

- Truthful play is an important benchmark in our empirical analysis

**Theorem**

*If students play truthfully the HBS mechanism is unit-trade ex-post efficient. That is, there do not exist Pareto-improving trades in which each student involved in the trade gives and receives one course.*

**Proof.** Suppose there exists a single-unit trade. Strictly order the set of students by the time they selected the course they are giving up in the trade. Call the first student \( i_1 \).

For the trade to be Pareto-improving it must be that \( c_{i_1}^{\text{get}} P_{i_1} c_{i_1}^{\text{give}} \). But \( c_{i_1}^{\text{get}} \) must have been available at the time he selected \( c_{i_1}^{\text{give}} \), so he must not have been playing truthfully, a contradiction.
Properties of the HBS Mechanism

Is it "Fair"?

- Fairness was described to us as the central concern of HBS in the design of their course-allocation procedure.

**Definition.** A choosing order $\lambda$ favors $s_1$ over $s_2$ if, for all $l \leq m$, $s_1$’s $l^{th}$ choice is earlier than $s_2$’s $l^{th}$ choice. A mechanism is **minimally fair** if, for every choosing order $\lambda$ it selects with positive probability, no one student is favored over any other student.

**Theorem**

Suppose $m = 2$. Then the HBS Mechanism is the unique anonymous choose-one-at-a-time mechanism that is minimally fair.

- For $m > 2$ there are multiple anonymous and minimally fair mechanisms.
- What would be the appropriate fairness concept in the Bidding Points class?
Empirical Analysis: Timing of Data

- May 1: poll of top 5 courses
- May 11: trial run of allocation mechanism
- Teaching evaluations for Winter courses released
- July: initial allocation
- Sept: Secondary market
- Jan: poll of top 30 courses

Full results released to students
Aggregate results released to students
Full results including ROLs 460 students
Full results of trial run including ROLs 165 students
Full results including ROLs 916 students
Full results 165 students

Our data:
Empirical Strategy

We have complete data on HBS students’ actual stated preferences (top 30), and incomplete data on (arguably) students’ truthful preferences (top 5, for 50% of students)

1. Support our claim that the top-5 survey data is truthful: students’ stated preferences differ from their true preferences in the manner predicted by the theory conjecture

2. Construct complete truthful preferences by merging truthful top 5 with strategic top 30 in a conservative way: congestion increases, as predicted by theory conjecture

3. Payoff I: Analyze effects of strategic play at the individual and aggregate level, by comparing HBS Truthful to HBS Strategic

4. Payoff II: Compare HBS Strategic versus Random Serial Dictatorship (which is strategyproof, so we know how students would play)
Evidence of strategic behavior

- Under the null hypothesis that changes in the preference submissions are only driven by idiosyncratic preference changes, the distribution of ranks in the May poll and the distribution of ranks in the July run should be the same.

- Statistical test: Extension of Mann-Whitney-Wilcoxon rank test due to Gehan (1965). Non parametric test for equality of distributions of discrete and censored data. One-sided test also available. (Gehan test, 5% significance)

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<td>Low demand courses</td>
<td>25</td>
<td>17</td>
<td>8</td>
<td>0</td>
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Evidence of strategic behavior (cont’d)

CDF of course ranks for selected high demand courses

May poll versus July ROL

Course 15XX

Course 12XX

Course 16XX

Course 21XX

Note: bootstrapped bounds, no correction for censoring
Evidence on increased congestion

Construction of Truthful Preferences

- For counterfactual, need to construct truthful preference lists that are longer than 5 courses.
- We assume the top 5 truthful courses correspond to top 5 courses in May poll. Other courses are moved down to position 6 and below in a way that preserves relative ordering of courses not in May poll ROL.
- Example:
  - May poll ROL: $c_1, c_2, c_3, c_4, c_5$
  - July run ROL: $c_4, c_3, c_6, c_1, c_2, c_7, c_8$
  - Constructed truthful preferences: $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$
- Provides a lower bound on extent of strategic behavior, since it assumes all courses not in top 5 were ranked truthfully.
Evidence on increased congestion (cont’d)

Comparison of sell-out times

- Over 5000 trials, we examine whether the course reaches capacity earlier / later under strategic play in >99% of orderings.

<table>
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<th>Later</th>
<th>Never</th>
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<td>16</td>
<td>1</td>
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<td>5</td>
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<td>Medium demand courses</td>
<td>37</td>
<td>1</td>
<td>3</td>
<td>17</td>
<td>16</td>
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<tr>
<td>Low demand courses</td>
<td>32</td>
<td>0</td>
<td>0</td>
<td>29</td>
<td>3</td>
</tr>
</tbody>
</table>

- All but 1 High-demand course reached capacity earlier on average, and most reached capacity earlier for all 5000 random orderings.

- For the 13 courses that reach capacity in the first 5 rounds under truthful play:
  - Average sell-out time under Truthful play: 3.02
  - Average sell-out time under Strategic play: 2.59
Summary of empirical evidence

- Students bias their submitted ROLs in favor of courses likely to reach capacity sooner
- Strategic behavior by students increases congestion
- Empirical evidence consistent with conjecture
No reason to expect Pareto-dominance relationship between HBS Truthful and HBS Strategic

Our first approach is to count strict ex-ante stochastic improvements at the individual level, as in Pathak (2006)

Challenge: data is ROLs over deterministic individual courses; outcomes are lotteries over bundles of courses

For Responsive preferences, a sufficient condition for $s$ to prefer Truthful play to Strategic play is if, for every $\lambda$, the student's outcome under Truthful play weakly dominates that under Strategic play

Example: $(1,0,0,1,0)$ dominates $(1,0,0,0,1)$ but does not dominate $(0,1,1,0,0)$
Welfare Consequences of Strategic Play: Individual Level
Comparison of HBS-Truthful to HBS-Strategic over 5000 trials

Outcome Preferences over 5000 Trials:
Truthful vs. Strategic Play, Assuming Responsive Preferences

- Strictly Prefer Truthful
- Strictly Prefer Strategic
- Strictly Indifferent
- Indeterminate
Our second approach is to look at the aggregate rank distributions - which represent social welfare under certain assumptions.
Since strategic behavior in the HBS mechanism harms welfare, it is natural to consider a strategyproof alternative.

We exploit the fact that the academic year has two semesters to create a more equitable ex-post form of RSD.

**Random Serial Dictatorship - Semesters.** Randomly order students.

- Fall courses: ascending order of priority number
- Spring courses: descending order of priority number

RSD-Sem is Minimally Fair, like the HBS mechanism.

RSD-Sem is Strategyproof.
HBS vs. RSD
Horserace between Two Externalities

- HBS suffers a negative *congestion* externality
- RSD suffers a negative *callousness* externality
  - Lucky students make their last choice independently of whether it would be an unlucky students’ first choice.

- We compare HBS and RSD-Semester using
  - Same measure as above. Mostly indeterminates.
  - Range of courses received: Best and Worst

- Caveat: we were conservative in construction of truthful preferences, so we likely underestimate the HBS congestion externality
HBS vs. RSD
Individual outcome comparisons over 5000 trials

- For Responsive preferences, our sufficient condition is so strong that the outcome becomes indeterminate for 97% of students.
- If we assume Additive preferences, we can make more comparisons:

![Graph showing outcome preferences over 5000 trials: RSD-Semesters vs. HBS-Strategic, assuming Additive Preferences. The graph indicates the percentage of trials for each category, with "Strictly Prefer RSD-Sem," "Strictly Prefer Strategic," "Strictly Indifferent," and "Indeterminate." The "Indeterminate" category shows the highest percentage.]
HBS vs. RSD
Best and Worst Received Course

CDF: Best and Worst Courses Received for HBS and RSD-Semesters

Cumulative Percentage

Rank of Best/Worst Received Course

- HBS
- RSD-Sem
Multi-unit Assignment is an important and unsolved problem
- Dictatorship is unacceptable to school administrations
- Any other Choose One at a Time mechanism: students overreport popular courses (Thm 2, Example 1)
- Bidding Points mechanism: students don't want to squander priority

Theory Conjecture: In equilibrium of the HBS mechanism, students overreport popular courses and congestion increases

Conjecture is confirmed empirically using a data set with both truthful and stated preferences

Strategic behavior harms welfare; magnitudes economically meaningful

Still, HBS mechanism appears preferable to RSD in ex-ante efficiency

Analysis cautions against strategyproofness as a starting point for market design in multi-unit assignment problems
RSD : Boston :: HBS : Stanford GSB?

- When asked why we’re inclined to prefer RSD to Boston, we can cite specific desirable properties of RSD that Boston lacks
  - Ex-Post Efficiency
  - Strategyproofness

- Intuition suggests that the HBS mechanism is preferable to the Stanford GSB mechanism for the same basic reason: no risk of squandering priority

- But, what properties can we cite?

- Is there a way to argue that HBS is a reasonable mechanism for course allocation from first principles, i.e. without the empirical analysis?

- Is there a way to improve on the HBS mechanism?