Math 2301320
Bi-invariant metrics on Lie groups

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1. The Lie algebra of a Lie group.
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The simplest example of a Riemann manifold is Euclidean space, where the geodesics are straight lines and all curvatures vanish. We may think of Euclidean space as a commutative Lie group under addition, and view the straight lines as translates of one parameter subgroups (lines through the origin). An easy but important generalization of this is when we consider bi-invariant metrics on a Lie group, a concept we shall explain below. In this case also, the geodesics are the translates of one parameter subgroups. Some of the material in today’s is a repeat or review of Problem set 1 and 2.
Definition of a Lie group

Let $G$ be a Lie group. This means that $G$ is a group, and is a smooth manifold such that the multiplication map $G \times G \to G$ is smooth, as is the map

$$\text{inv} : G \to G$$

sending every element into its inverse:

$$\text{inv} : a \mapsto a^{-1}, \quad a \in G.$$  

Until now the Lie groups we studied were given as subgroups of $GL(n)$. We can continue in this vein, or work with the more general definition just given. We have the left action of $G$ on itself

$$L_a : G \to G, \quad b \mapsto ab$$

and the right action

$$R_a : G \to G, \quad b \mapsto ba^{-1}.$$  

We let $g$ denote the tangent space to $G$ at the identity:

$$g = TG.$$  

For the convenience of the reader, I now review and expand upon the discussion in Problem set 2:
If \( v \in TG_a \) is tangent vector at the point \( a \in G \), there will be a unique left invariant vector field \( X \) such that \( X(a) = v \). In other words, there is a linear map

\[
\omega_a : TG_a \rightarrow g
\]

sending the tangent vector \( v \) to the element \( \xi = \omega_a(v) \in g \) where the left invariant vector field \( X \) corresponding to \( \xi \) satisfies \( X(a) = v \). So we have defined a \( g \) valued linear differential form \( \omega \) identifying the tangent space at any \( a \in G \) with \( g \). If

\[
dL_b v = w \in TG_{ba}
\]

then \( X(ba) = w \) since \( X(a) = v \) and \( X \) is left invariant. In other words,
\[ \omega_{L_b a} \circ dL_b = \omega_a, \]
or, what amounts to the same thing

\[ L_b^* \omega = \omega \]

for all \( b \in G \). The form \( \omega \) is left invariant. When we proved this for a subgroup of \( Gl(n) \) this was a computation. But in the general case, as we have just seen, it is a tautology. We now want to establish the generalization of the Maurer-Cartan equation which said that for subgroups of \( Gl(n) \) we have

\[ d\omega + \omega \wedge \omega = 0. \]
Since we no longer have, in general, the notion of matrix multiplication which enters into the definition of $\omega \wedge \omega$, we must first must rewrite $\omega \wedge \omega$ in a form which generalizes to an arbitrary Lie group. So let us temporarily consider the case of a subgroup of $GL(n)$. Recall that for any two form $\tau$ and a pair of vector fields $X$ and $Y$ we write $\tau(X, Y) = i(Y)i(X)\tau$. Thus

$$(\omega \wedge \omega)(X, Y) = \omega(X)\omega(Y) - \omega(Y)\omega(X),$$

the commutator of the two matrix valued functions, $\omega(X)$ and $\omega(Y)$.
Consider the commutator of two matrix valued one forms, $\omega$ and $\sigma$,

$$\omega \wedge \sigma + \sigma \wedge \omega$$

(according to our usual rules of superalgebra). We denote this by

$$[\omega \wedge, \sigma].$$

In particular we may take $\omega = \sigma$ to obtain

$$[\omega \wedge, \omega] = 2\omega \wedge \omega.$$
So we can rewrite the Maurer-Cartan equation for a subgroup of \( \text{GL}(n) \) as

\[
    d\omega + \frac{1}{2} [\omega \wedge, \omega] = 0. \tag{5}
\]

Now for a general Lie group we \textit{do} have the Lie bracket map

\[
    \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}.
\]

So we can define the two form \([\omega \wedge, \omega]\). It is a \( \mathfrak{g} \) valued two form which satisfies

\[
    i(X)[\omega \wedge, \omega] = [X, \omega] - [\omega, X]
\]

for any left invariant vector field \( X \). Hence

\[
    [\omega \wedge, \omega](X, Y) := i(Y)i(X)[\omega \wedge, \omega] = i(Y)([X, \omega] - [\omega, X])
\]

\[
    = [X, Y] - [Y, X] = 2[X, Y]
\]

for any pair of left invariant vector fields \( X \) and \( Y \).
So to prove (5) in general, we must verify that for any pair of left invariant vector fields we have

\[ d\omega(X, Y) = -\omega([X, Y]). \]

But this is a consequence of our general formula for the exterior derivative which in our case says that

\[ d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]). \]

In our situation the first two terms on the right vanish since, for example, \( \omega(Y) = Y = \eta \) a constant element of \( g \) so that \( X\omega(Y) = 0 \) and similarly \( Y\omega(X) = 0 \).
Invariant metrics

Any non-degenerate scalar product, $\langle \cdot, \cdot \rangle$, on $\mathfrak{g}$ determines (and is equivalent to) a left invariant semi-Riemann metric on $G$ via the left-identification $dL_a : \mathfrak{g} = T_{g_e} G \rightarrow T_{g_a} G, \ \forall \ a \in G$.

Since $A_a = L_a \circ R_a$, the left invariant metric, $\langle \cdot, \cdot \rangle$ is right invariant if and only if it is $A_a$ invariant for all $a \in G$, which is the same as saying that $\langle \cdot, \cdot \rangle$ is invariant under the adjoint representation of $G$ on $\mathfrak{g}$, i.e. that

$$\langle Ad_a Y, Ad_a Z \rangle = \langle Y, Z \rangle, \ \forall Y, Z \in \mathfrak{g}, \ a \in G.$$
Setting $a = \exp tX$, $X \in \mathfrak{g}$, differentiating with respect to $t$ and setting $t = 0$ gives

$$\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0, \quad \forall X, Y, Z \in \mathfrak{g}.$$  \hfill (6)

If every element of $G$ can be written as a product of elements of the form $\exp \xi$, $\xi \in \mathfrak{g}$ (which will be the case if $G$ is connected), this condition implies that $\langle \cdot, \cdot \rangle$ is invariant under $Ad$ and hence is invariant under right and left multiplication. Such a metric is called **bi-invariant**.
Recall that \( \text{inv} \) denotes the map sending every element into its inverse:

\[
\text{inv} : a \mapsto a^{-1}, \quad a \in G.
\]

Since \( \text{inv} \exp tX = \exp(-tX) \) we see that

\[
d \text{inv}_e = -\text{id}.
\]

Also

\[
\text{inv} = R_a \circ \text{inv} \circ L_{a^{-1}}
\]

since the right hand side sends \( b \in G \) into

\[
b \mapsto a^{-1}b \mapsto b^{-1}a \mapsto b^{-1}.
\]
Hence $d \text{ inv}_a : TG_a \rightarrow TG_{a^{-1}}$ is given, by the chain rule, as

$$dR_a \circ \text{ inv}_e \circ dL_{a^{-1}} = -dR_a \circ dL_{a^{-1}}$$

implying that a bi-invariant metric is invariant under the map inv. Conversely, if a left invariant metric is invariant under inv then it is also right invariant, hence bi-invariant since

$$R_a = \text{ inv} \circ L_a \circ \text{ inv}.$$
Let $G \times G$ act on $G$ by left and right multiplication, i.e. $(a, b) \in G \times G$ sends $c \in G \mapsto acb^{-1}$. Denote this action by $\kappa$. So a metric $g$ on $G$ is bi-invariant if and only if $\kappa(a, b)^*g$ for all $(a, b) \in G \times G$.

If $G$, and hence $G \times G$ is compact, then there is an invariant integral $\mu$ on $G \times G$, and for any Riemannian metric $g$, the metric

$$h := \int_{G \times G} \kappa(a, b)^*g \, d\mu(a, b)$$

is invariant.
The Koszul formula for bi-invariant metrics

The Koszul formula simplifies considerably when applied to left invariant vector fields and bi-invariant metrics since all scalar products are constant, so their derivatives vanish, and we are left with

\[ 2\langle \nabla_X Y, Z \rangle = -\langle X, [Y, Z] \rangle - \langle Y, [X, Z] \rangle + \langle Z, [X, Y] \rangle \]

and the first two terms cancel by (6). We are left with

\[ \nabla_X Y = \frac{1}{2} [X, Y]. \] (7)
Conversely, if \( \langle \ , \ \rangle \) is a left invariant metric for which

\[ \nabla_X Y = \frac{1}{2} [X, Y] \quad (7) \]

holds, then

\[
\begin{align*}
\langle X, [Y, Z] \rangle &= 2 \langle X, \nabla Y Z \rangle \\
&= -2 \langle \nabla Y X, Z \rangle \\
&= -\langle [Y, X], Z \rangle \\
&= \langle [X, Y], Z \rangle
\end{align*}
\]

so the metric is bi-invariant.
Geodesics are cosets of one parameter groups.

Let $\alpha$ be an integral curve of the left invariant vector field $X$. Equation

$$\nabla_X Y = \frac{1}{2} [X, Y] \quad (7)$$

implies that $\alpha'' = \nabla_X X = 0$ so $\alpha$ is a geodesic. Thus the one-parameter groups are the geodesics through the identity, and all geodesics are left cosets of one parameter groups. (This is the reason for the name exponential map in the theory of linear connections.)
We compute the Riemann curvature of a bi-invariant metric by applying its definition to left invariant vector fields:

$$R_{XY}Z = \frac{1}{4}[X, [Y, Z]] - \frac{1}{4}[Y, [X, Z]] - \frac{1}{2}[[X, Y], Z]$$

Jacobi’s identity implies the first two terms add up to $\frac{1}{4}[[X, Y], Z]$ and so

$$R_{XY}Z = -\frac{1}{4}[[X, Y], Z].$$

(8)
Sectional curvatures.

In particular

\[ \langle R_{XY} X, Y \rangle = -\frac{1}{4} \langle [[X, Y], X], Y \rangle = -\frac{1}{4} \langle [X, Y], [X, Y] \rangle \]

so the sectional curvature is given by

\[ K(X, Y) = \frac{1}{4} \frac{||[X, Y]||^2}{||X \wedge Y||^2}. \quad (9) \]

Notice that in the Riemannian case (but not, in general, in the semi-Riemannian case) this expression is non-negative.
Recall that for each $X \in g$ the linear transformation of $g$ consisting of bracketing on the left by $X$ is called $\text{ad } X$. So

$$\text{ad } X : g \to g, \quad \text{ad } X(V) := [X, V].$$

We can thus write our formula for the curvature as

$$R_{XV} Y = \frac{1}{4} (\text{ad } Y)(\text{ad } X)V.$$

Now the Ricci curvature was defined as

$$\text{Ric } (X, Y) = \text{tr } [V \mapsto R_{XV} Y].$$
We thus see that for any bi-invariant metric, the Ricci curvature is always given by

$$\text{Ric} = \frac{1}{4} B$$

(10)

where $B$, the **Killing form**, is defined by

$$B(X, Y) := \text{tr} \ (\text{ad} \ X)(\text{ad} \ Y).$$

(11)
The Killing form is symmetric, since $\text{tr} \ (AC) = \text{tr} \ CA$ for any pair of linear operators. It is also invariant. Indeed, let $\mu : \mathfrak{g} \to \mathfrak{g}$ be any automorphism of $\mathfrak{g}$, so $\mu([X, Y]) = [\mu(X), \mu(Y)]$ for all $X, Y \in \mathfrak{g}$. We can read this equation as saying

$$\text{ad} \ (\mu(X))(\mu(Y)) = \mu(\text{ad}(X)(Y))$$

or

$$\text{ad} \ (\mu(X)) = \mu \circ \text{ad} \ X \mu^{-1}.$$  

Hence

$$\text{ad} \ (\mu(X))\text{ad} \ (\mu(Y)) = \mu \circ \text{ad} \ X \text{ad} \ Y \mu^{-1}.$$  

Since trace is invariant under conjugation, it follows that

$$B(\mu(X), \mu(Y)) = B(X, Y).$$
\[ B(\mu(X), \mu(Y)) = B(X, Y). \]

Applied to \( \mu = \exp(tad \, Z) \) and differentiating at \( t = 0 \) shows that

\[ B([Z, X], Y) + B(X, [Z, Y]) = 0. \]
So the Killing form defines a bi-invariant symmetric bilinear form on $G$. Of course it need not, in general, be non-degenerate. For example, if the group is commutative, it vanishes identically. A group $G$ is called **semi-simple** if its Killing form is non-degenerate. So on a semi-simple Lie group, we can always choose the Killing form as the bi-invariant metric. For such a choice, our formula above for the Ricci curvature then shows that the group manifold with this metric is **Einstein**, i.e. the Ricci curvature is a multiple of the scalar product.

Suppose that the adjoint representation of $G$ on $\mathfrak{g}$ is irreducible. Then $\mathfrak{g}$ can not have two invariant non-degenerate scalar products unless one is a multiple of the other. In this case, we can also conclude from our formula that the group manifold is Einstein.
Here is a way to construct invariant scalar products on a Lie algebra \( \mathfrak{g} \) of a Lie group \( G \). Let \( \rho \) be a representation of \( G \). This means that \( \rho \) is a smooth homomorphism of \( G \) into \( Gl(n, \mathbb{R}) \) or \( Gl(n, \mathbb{C}) \). This induces a representation \( \dot{\rho} \) of \( \mathfrak{g} \) by

\[
\dot{\rho}(X) := \frac{d}{dt} \rho(\exp tX)|_{t=0}.
\]

So

\[
\dot{\rho} : \mathfrak{g} \to gl(n)
\]

where \( gl(n) \) is the Lie algebra of \( Gl(n) \), and

\[
\dot{\rho}([X, Y]) = [\dot{\rho}(X), \dot{\rho}(Y)]
\]

where the bracket on the right is in \( gl(n) \). More generally, a linear map \( \dot{\rho} : \mathfrak{g} \to gl(n, \mathbb{C}) \) or \( gl(n, \mathbb{R}) \) satisfying the above identity is called a representation of the Lie algebra \( \mathfrak{g} \). Every representation of \( G \) gives rise to a representation of \( \mathfrak{g} \) but not every representation of \( \mathfrak{g} \) need come from a representation of \( G \) in general.
If $\dot{\rho}$ is a representation of $\mathfrak{g}$, with values in $gl(n, \mathbb{R})$, we may define

$$\langle X, Y \rangle_{\mathfrak{g}} := \text{tr} \dot{\rho}(X)\dot{\rho}(Y).$$

This is real valued, symmetric in $X$ and $Y$, and

$$\langle [X, Y], Z \rangle_{\mathfrak{g}} + \langle Y, X, Z \rangle_{\mathfrak{g}} =$$

$$\text{tr} (\dot{\rho}(X)\dot{\rho}(Y)\dot{\rho}(Z) - \dot{\rho}(Y)\dot{\rho}(X)\dot{\rho}(Z)\dot{\rho}(Y)\dot{\rho}(X)\dot{\rho}(Z) - \dot{\rho}(Y)\dot{\rho}(Z)\dot{\rho}(X))$$

$$= 0.$$

So this is invariant. Of course it need not be non-degenerate.
A case of particular interest is when the representation $\dot{\rho}$ takes values in $u(n)$, the Lie algebra of the unitary group. An element of $u(n)$ is a skew adjoint matrix, i.e. a matrix of the form $iA$ where $A = A^*$ is self adjoint. If $A = A^*$ and $A = (a_{ij})$ then

$$
\text{tr} A^2 = \text{tr} AA^* = \sum_{i,j} a_{ij}a_{ji} = \sum_{i,j} a_{ij}\overline{a_{ij}} = \sum_{ij} |a_{ij}|^2
$$

which is positive unless $A = 0$. So

$$
- \text{tr}(iA)(iA)
$$

is positive unless $A = 0$. This implies that if $\dot{\rho} : g \to u(n)$ is injective, then the form

$$
\langle X, Y \rangle = - \text{tr} \dot{\rho}(X)\dot{\rho}(Y)
$$

is a positive definite invariant scalar product on $g$. 