1. Affine geometry

Much of this material is taken from Hartshorne [1].

Definition 1.1. An Affine plane is a set $A$ whose elements are called points and a collection of subsets of $A$ called “lines” satisfying the properties listed below.

Now slow down there. Before stating the properties we need less fancy terminology. If $\ell$ is a line and $P$ is a point, we will use the colloquialisms “$\ell$ contains $P$” or “$P$ lies on $\ell$” to indicate that $P$ is an element of the set $\ell$. We will also say that a set of points $\{P_1, \ldots, P_n\}$ is collinear if there is a line $\ell$ containing them all. Finally two lines $\ell_1$ and $\ell_2$ are parallel if they are the same line ($\ell_1 = \ell_2$), or if they have no points in common ($\ell_1 \cap \ell_2 = \emptyset$).

A1. Given two distinct points $P$ and $Q$ there is a unique line $\ell$ containing them both.

A2. Given a line $\ell$ and a point $P$, there is one and only one line $m$ which is parallel to $\ell$ and passes through $P$.

A3. There exist three non-collinear points.

Before doing anything with this, let’s set up some examples. Let’s first check that the thing we’re most familiar with is one.

Definition 1.2. The Euclidean plane is the affine geometry whose points are pairs $(x, y)$ of real numbers, and whose lines are the sets of points of the form

$$\ell_{a,b,c} = \{(x, y) \mid ax + by + c = 0\}$$

in which at least one of $\{a, b, c\}$ is non-zero.

Note that the lines $\ell_{a,b,c}$ and $\ell_{a',b',c'}$ coincide if and only if there is a non-zero real number $\lambda$ with

$$a' = \lambda a \quad b' = \lambda b \quad c' = \lambda c.$$

We need to check that this data defines an affine plane. Just to give you an idea, let’s prove A1. Once you have the idea, you’ll be able to deal with the other axioms. Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are two distinct points. The fact that they are distinct means that one of $x_1 - x_2$ and $y_1 - y_2$ is non-zero. Suppose it is $x_1 - x_2$. We need to find a line $\ell_{a,b,c}$ containing both $P$ and $Q$, and we need to show the line is unique. Expanding out the definition, this means that we need to solve the equations

$$ax_1 + by_1 + c = 0$$
$$ax_2 + by_2 + c = 0$$
for $a$, $b$, and $c$, and we need to show that the solution is unique up to multiplication by $\lambda$. This is a simple problem in linear algebra. You ought to do this by row-reducing. If you do, you get the system of equations

$$ax_1 + by_1 + c = 0$$
$$a + b(y_2 - y_1)/(x_2 - x_1) + = 0.$$

Given any $b$, the second equation determines $a$, and then the first determines $c$. Replacing $b$ by $\lambda b$ replaces $a$ by $\lambda a$ and $c$ by $\lambda c$. This proves axiom A1. In the exercise you will handle the other two exercises.

Are there any other examples? Let’s try and write down the smallest possible. Before doing so, it will be helpful to point out a lemma which will be useful later.

**Lemma 1.3.** In an affine plane, the relation “parallel” is an equivalence relation.

**Proof:** The symmetry relation $\ell \parallel \ell$ holds by definition (this is the reason we allow the case $\ell_1 = \ell_2$ in the definition of parallel.) The reflexive property is also immediate from the definition. The real work is in proving that the relation is transitive. So suppose $\ell_1 \parallel \ell_2$ and $\ell_2 \parallel \ell_3$. If $\ell_1$ and $\ell_3$ have no points in common, they are parallel, and we are done. So suppose they both contain a point $P$. Then $\ell_1$ and $\ell_3$ contain $P$ and are parallel to $\ell_2$. By the uniqueness in A2, this means that $\ell_1 = \ell_3$ and they are parallel. □

On to our tiny plane. We know by A3 that there are at least three points. Let’s call them $P$, $Q$, and $R$. The point $R$ is not on the line $PQ$, and the point $P$ is not on $QR$. By A2 there is a line $\ell_R$ containing $R$ and parallel to $PQ$ and a line $\ell_Q$ containing $Q$ and parallel to $PR$.

![Diagram 1.4](image)

Now $\ell_Q$ and $\ell_R$ cannot be parallel. If the were we would have

$$PR \parallel \ell_Q \parallel \ell_R \parallel PQ$$

and so $PQ$ and $PR$ would also be parallel by Lemma 1.3. But $PR$ and $PQ$ are not parallel (why?). This means that there is a point $S$ lying on both $\ell_Q$ and $\ell_R$. This point cannot coincide with $P$, $Q$, or $R$, since each of these points lies on exactly one of $\ell_Q$ or $\ell_R$. Thus $S$ is a unique fourth point. This shows that an affine plane must contain at least four points. The diagram (1.4) exhibits an affine plane with exactly four points, and exactly 6 lines.

In the picture, the two lines which appear to cross ($PS$ and $QR$) are in fact parallel. In a way it is better to think of the picture as displayed in Figure 1. In that picture the line $QR$ and $PS$ actually look parallel.

We can try to define an $n \times n$ affine plane in the same manner. I will leave the details to you for now. Figure 2 shows the 9-point $3 \times 3$ plane, and a few lines in it. We will see later that the $n \times n$ plane satisfies the axioms for an affine plane if and only if $n$ is a prime number.

Note that all the lines in the 4-point plane have exactly 2 points, and all of the lines in the 9-point affine plane have exactly 3 points. This is not an accident.
Proposition 1.5. There is a bijection between the points of any two lines in an affine plane.

Proof: Suppose that $\ell_1$ and $\ell_2$ are two distinct lines. Then there is a point $P_1$ on $\ell_1$ which is not on $\ell_2$, and a point $P_2$ on $\ell_2$ which is not on $\ell_1$. Write $\ell = P_1P_2$. We may now set up our one-to-one correspondence. Suppose that $X$ is any point on $\ell_1$. By A2 there is a unique line through $X$ parallel to $\ell$. This line cannot be parallel to $\ell_2$ (why?), and so meets $\ell_2$ in a unique point $Y$. I leave it to you to check that this correspondence between points of $\ell_1$ and points of $\ell_2$ is a bijection. See Figure 3 for a picture. $\square$

Families of lines come up a lot in affine geometry, and it is useful to have a name for the kind that do.
Definition 1.6. The set of lines through a fixed point $p$ in an affine plane is called the \textit{pencil of lines through} $p$. The set of lines parallel to a fixed line $\ell$ the \textit{pencil of lines parallel to} $\ell$.

Exercises

1.1. Prove A2 for the Euclidean plane as follows. Suppose that $\ell_{a,b,c}$ is a line, and $P = (x, y)$ is a point not on $\ell$. Let $c' = -ax - by$. Show that the point $P$ lies on the line $\ell_{a,b,c'}$, and that the lines $\ell_{a,b,c}$ and $\ell_{a,b,c'}$ are parallel.

1.2. Prove A3 for the Euclidean plane. Make sure you really prove it.

1.3. Let $A = \mathbb{R}^2 \setminus \{0\}$ be the complement of the origin in the Euclidean plane, and define a line in $A$ to be a subset of the form $\ell \cap A$, where $\ell \subset \mathbb{R}^2$ is a Euclidean line. Does this define an affine plane?

1.4. Show that the $4 \times 4$-plane is not an affine plane.

1.5. Suppose that $\ell$ is a line in an affine plane. Show that there is another line $\ell' \neq \ell$ which is parallel to $\ell$. Show that there is another line $\ell'' \neq \ell$ which is not parallel to $\ell$.

1.6. Show that every line in an affine plane contains at least two points.

1.7. In this exercise we set up coordinate systems. The construction is illustrated in Figure 4. Suppose that $A$ is an affine plane. Choose a line $\ell_1$ and another line $\ell_2$ which is not parallel to $\ell_1$. Given a point $P$ show that the unique line parallel to $\ell_2$ and containing $P$ cannot be parallel to $\ell_1$. Let $x \in \ell_1$ be the unique point of intersection. Similarly, let $y \in \ell_2$ be the unique point at which the line parallel to $\ell_1$ and containing $P$ meets $\ell_2$. Show that the correspondence

$$A \to \ell_1 \times \ell_2$$

$$P \mapsto (x, y)$$

is a bijection. Conclude that if an affine plane has one line with $n < \infty$ points, then it is a finite affine plane and has $n^2$ points.

1.8. Show that an affine plane with $n^2$ points has exactly $n^2 + n$ lines.
2. Projective geometry

Things work out a little cleaner if we extend our affine planes by adding “ideal points” so that even parallel lines intersect.

Definition 2.1. A projective plane is a set \( P \) called the set of “points” and a collection of subsets of \( P \) called “lines” satisfying \( P1-P4 \) below.

- **P1.** Given two distinct points, there is a unique line containing them both.
- **P2.** Any two lines meet in at least one point.
- **P3.** There exist three non-collinear points.
- **P4.** Every line contains at least three points.

A little explanation is in order. When \( P \) and \( Q \) are two distinct points it is common to use the symbol \( PQ \) to denote the unique line, guaranteed by \( P1 \), containing them both. As for \( P2 \), suppose that \( \ell_1 \) and \( \ell_2 \) are two lines. By \( P2 \) they meet in at least one point. If they meet in two points, they are the same line by \( P1 \). So \( P2 \) could have been written as “two distinct lines meet in exactly one point.”

**Example 2.2.** Let \( A \) be an affine plane, and \( \ell \) a line in \( A \). The ideal point of \( \ell \) is the equivalence class \([\ell]\) of all lines parallel to \( \ell \) (or, just to use our new language, the pencil of lines parallel to \( \ell \)). We define the extended affine plane \( P \) to be the set whose elements are \( A \) and the ideal points \([\ell]\) of the lines of \( A \). A line in \( P \) is either an extended line of \( A \), which means a set whose elements consist of the points of a line \( \ell \) and the ideal point \([\ell]\), or the line at infinity which means the set consisting of all the ideal points \([\ell]\). The ideal point \([\ell]\) is also called the point at infinity of \( \ell \). With this structure the extended affine plane \( P \) is a projective plane. Two parallel lines \( \ell_1 \) and \( \ell_2 \) of \( A \) now meet at their ideal points, and every line meets the line at infinity at its ideal point. The projective extension of the 4-point plane is illustrated in Figure 5. It is sometimes called the Fano plane.

**Example 2.3.** There is another way to get a projective plane from an affine plane \( A \). Let \( P^* \) be the set of lines in \( A \), and define a line in \( P^* \) to be a pencil of lines in \( A \). I leave it to you to check that this data defines a projective plane. I really do. See the exercises.
Example 2.4. The Euclidean projective plane or real projective plane, $\mathbb{RP}^2$ is the extended Euclidean plane. It can be described in terms of coordinates as follows. The set of points of $\mathbb{RP}^2$ is the set of all equivalences classes $[x, y, z]$ of triples of real numbers, at least one of which is non-zero. The equivalence relation is that $[x, y, z] = [\lambda x, \lambda y, \lambda z]$ whenever $\lambda$ is a non-zero real number. The lines are the sets of the form $\ell_{a,b,c} = \{[x, y, z] \mid ax + by + cz = 0\}$, in which at least one of $\{a, b, c\}$ is non-zero. Note that $\ell_{a,b,c} = \ell_{a',b',c'}$ if and only if there is a non-zero $\lambda$ for which $a' = \lambda a$, $b' = \lambda b$, and $c' = \lambda c$.

The points $[x, y, z]$ with $z \neq 0$ are called the finite points and the points $[x, y, 0]$ form the projective line at infinity. This corresponds to the identification of the real projective plane with the extended Euclidean plane. Indeed, the set of finite points can be identified with the Euclidean plane by sending $[x, y, z]$ to $(x/z, y/z)$ and sending $(x, y)$ to $[x, y, 1]$. The ideal point of a Euclidean line $ax + by + c = 0$ is $[a, b, 0]$.

One way to think of the real projective plane is in terms of perspective drawing. Suppose you are sitting at the origin in $\mathbb{R}^3$, and your canvas is the plane $z = 1$. When you see a point $(x, y, z)$ you want to draw, you make a mark on your canvas at the point where the line through $(x, y, z)$ and the origin meets the plane $z = 1$. Thus the points on your canvas really correspond to lines through the origin in $\mathbb{R}^3$. I discussed this point of view further in class.

For our discussion of Desargues theorem, we will need the notion of a projective 3-space.

Definition 2.5. A projective 3-space is a set whose elements are called points, together with certain subsets called lines, and certain other subsets called planes, which satisfies the following axioms:

- S1 Two distinct points lie on a unique line.
- S2 Three non-collinear points lie on a unique plane.
- S3 A line meets a plane in at least one point.
- S4 Two planes have at least a line in common.
- S5 There exist four non-coplanar points, no three of which are collinear.
Every line has at least three points.

**Example 2.6.** The Euclidean projective 3-space $\mathbb{RP}^3$ is the set of lines through the origin in $\mathbb{R}^4$. Its points are equivalences classes $[x, y, z, w]$ of 4-tuples of real numbers. The equivalence relation is such that $[x, y, z, w] = [\lambda x, \lambda y, \lambda z, \lambda w]$, with $\lambda \neq 0 \in \mathbb{R}$. A line is a set of points of the form

$$\ell_T = \{[x, y, z, w] \mid [x, y, z, w] \cdot T = 0\},$$

where $T$ is a $4 \times 2$ matrix with the property that one of its $2 \times 2$ sub-matrices has a non-zero determinant. A plane is a set of points of the form

$$P_{a,b,c,d} = \{[x, y, z, w] \mid ax + by + cz + dw = 0\},$$

in which at least one of $a$, $b$, $c$, or $d$ is non-zero. We could write this last condition as $[x, y, z, w] \cdot L = 0$, with $L$ a non-zero $4 \times 1$ matrix. In more geometric terms, a point of $\mathbb{RP}^3$ is a line through the origin in $\mathbb{R}^4$, a line of $\mathbb{RP}^3$ is a 2-plane through the origin in $\mathbb{R}^4$, and a plane of $\mathbb{RP}^3$ is a 3-plane through the origin in $\mathbb{R}^4$.

**Exercises**

1. Show that there is a bijection between any two lines in a projective plane.

2. Suppose that $P$ is a projective plane. Let $P^*$ be the set of lines of $P$, and define a line of $P^*$ to be a pencil of lines in $P$. Show that with this structure, $P^*$ is a projective plane.

3. Let $\ell$ be a line in a projective plane, and $P$ a point not on $\ell$. Show that the map which associates to a point $Q$ of $\ell$ the line $PQ$ gives a bijection between the points of $\ell$ and the pencil of lines through $P$.

4. Show that if one line of a projective plane $P$ has $(n + 1)$-points, then $P$ is finite and has $n^2 + n + 1$ points. Show that every pencil of lines in $P$ contains exactly $(n + 1)$ lines. Show that $P$ has $n^2 + n + 1$ lines.

5. Suppose that $S$ is a projective 3-space. Show that a line and a plane which does not contain the line meet in exactly one point, and that two distinct planes in $S$ meet in exactly one line.

6. Suppose that $S$ is a projective 3-space. Let $S^*$ be the set of planes in $S$. Define a line in $S^*$ to be a pencil of planes through a line of $S$, and define a plane of $S^*$ to be a set of planes in $S$ passing through a given point of $S$. Thus there are bijections

$$\text{points of } S^* \leftrightarrow \text{planes of } S,$$

$$\text{lines of } S^* \leftrightarrow \text{lines of } S,$$

$$\text{planes of } S^* \leftrightarrow \text{points of } S.$$

Show that with this structure $S^*$ is a projective 3-space.

**References**


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